

THE CONTINUOUS SPIN REPRESENTATIONS OF THE POINCARÉ AND
SUPER-POINCARÉ GROUPS AND THEIR CONSTRUCTION
BY THE INÖNÜ-WIGNER GROUP CONTRACTION

By

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To my mother and my late father

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THE CONTINUOUS SPIN REPRESENTATIONS OF THE POINCARÉ AND
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We study the continuous spin representation (CSR) of the Poincaré group in arbitrary dimensions. In d dimensions, the CSRs are characterized by the length of the light-cone vector and the Dynkin labels of the $SO(d-3)$ short little group which leaves the light-cone vector invariant. In addition to these, a solid angle Ω_{d-3} which specifies the direction of the light-cone vector is also required to label the states. We also find supersymmetric generalizations of the CSRs. In four dimensions, the supermultiplet contains one bosonic and one fermionic CSRs which transform into each other under the action of the supercharges. In a five dimensional case, the supermultiplet contains two bosonic and two fermionic CSRs which is like $N = 2$ supersymmetry in four dimensions. When constructed using Grassmann parameters, the light-cone vector becomes nilpotent. This makes the representation finite dimensional, but at the expense of introducing central charges even though the representation is massless. This leads to zero or negative norm states. The nilpotent constructions are valid only for even dimensions.

We also show how the CSRs in four dimensions can be obtained from five dimensions by the combinations of Kaluza-Klein(KK) dimensional reduction and the Inönü-Wigner group contraction. The group contraction is a singular transformation. We show that the group contraction is equivalent to imposing periodic boundary condition along one direction and taking a double singular limit. In this form the contraction parameter is interpreted as the inverse KK radius. We apply this technique to both five dimensional regular massless and massive representations. For the regular massless case, we find that the contraction gives the CSR in four dimensions under a double singular limit and the representation wavefunction is the Bessel function. For the massive case, we use Majorana's

infinite component theory as a model for the $SO(4)$ little group. In this case, a triple singular limit is required to yield any CSR in four dimensions. The representation wavefunction is the Bessel function, as expected, but the scale factor is not the length of the light-cone vector. The amplitude and the scale factor are implicit functions of the parameter y which is a ratio of the internal and external coordinates. We also state under what conditions our solutions become identical to Wigner's solution.

CHAPTER 1 INTRODUCTION

All elementary particles in nature are labelled by their masses and spins or helicities. The underlying mathematical principle for this characterization is the representations of the Poincaré group. In 1939, E. P. Wigner first classified all representations of the four dimensional Poincaré group in a classic paper [1]. These classifications are characterized by two Casimir eigenvalues: the mass-squared M^2 and the square of the Pauli-Lubanski vector W^2 . The orthogonality between the Pauli-Lubanski vector and the four-momentum vector defines the little group which leaves the momentum vector invariant. The little group gives the spin (helicity) of the massive (massless) particles respectively. We summarize the classification in the following:

Representation	First Casimir	Second Casimir	Little Group
Massive representation	$M^2, M \text{ real}$	$W^2 = -M^2 j(j+1)$	$SO(3)$
Regular massless representation	$M = 0$	$W^2 = 0$	$SO(2)$
Tachyonic representation	$M^2, M \text{ imaginary}$	$W^2 = -M^2 j(j+1)$	$SO(3)$
Continuous spin representation	$M = 0$	$W^2 = \Xi^2$	$E(2)$

Among these four, only the first two representations are realized in nature. The massive particles, like protons, neutrons, π -mesons, etc., belong to the massive representations of different spin, and the massless particles like photons are described by the regular massless representations. Tachyons have imaginary mass, and violate causality as they travel faster than the speed of light [2]. They are unphysical and do not occur in nature. In field theory, the appearance of the tachyonic mass term is removed by shifting the vacuum, a process known as the Higgs mechanism.

The continuous spin representation (CSR), which is the main subject in this dissertation, is characterized by the zero mass and non-zero finite length of the Pauli-Lubanski vector. In four dimensions, the Pauli-Lubanski vectors are related to the translation generators which together with the homogeneous rotation generator form the inhomogeneous little group. Group theoretically this is the $E(2)$ algebra of the little group. In this representation, an action of a finite boost on any state produces an infinite tower of massless states with helicities ranging from minus to plus infinity. Even though the CSR is mathematically very sound, there are various arguments against

its existence in nature. Wigner [3] himself argued that since the CSR gives an infinite heat capacity of the vacuum, it is unphysical and does not exist in nature. L. Abbott [4] showed that the CSR violates causality and K. Hirata [5] further showed that in this representation the fields become non-local. Moreover, S. Weinberg and E. Witten [6] proved that in four dimensions there cannot exist any particle with helicity higher than two. Because of these strong arguments, the CSR has never attracted much attention.

Despite these facts, there are still reasons for further studying the CSR, in particular in the context of models for extended objects in higher dimensions, as for example, String theory and M-theory. According to these models, the infinite slope limit of the Regge trajectories yields an infinite helicity tower of massless particles. Mathematically this is the representation of the Euclidean group which is the little group of the Poincaré group.

This is our main motivation to study the CSR. We will not address how the problems associated with the CSR can be cured, but will concentrate on their mathematical properties. We hope that these studies may provide better insights for how to avoid or cure the problems associated with the CSR. Our aim is two-fold. First, we study how these representations and their states can be characterized in any dimensions by the Casimirs of the little groups and if these representations have any supersymmetric extension. Because of the supersymmetry, the light-cone vector can be constructed with or without using the Grassmann parameters. This means the light-cone can be made nilpotent and as a consequence the representation can become finite-dimensional.

Second, we study how these representations can be obtained from larger groups. Of particular interest in physics are the Conformal groups which contain the Poincaré group by embedding and the larger Poincaré group, (Anti) de Sitter groups which give the Poincaré group by group contraction only. It has been known for sometime that the CSR has no conformal extension [7, 8]. We prove this in light-cone language in Appendix A. Therefore, we will look into group contractions.

Group contraction has been introduced by İnönü and Wigner [9] in 1953, and independently by Segal [10]. The group contraction is a singular transformation to obtain the Euclidean group from a homogeneous group. İnönü and Wigner (IW) showed explicitly how the contraction of the $SO(3)$ group yields the $E(2)$ group which is algebraically identical to the Poincaré little group in four dimensions. We show that the IW group contraction can be realized in a different way which is more physical. Imposing a periodic boundary condition and taking a double singular limit yield the same conclusion as IW. We consider the periodic boundary condition as the Kaluza-Klein(KK) dimensional reduction process. One singular limit can be identified with the infinite KK radius

limit whose inverse is the contraction parameter and the other singular limit is the spin. We apply this technique to both regular massless and massive representations in five dimensions. We find the conditions for the CSR to exist in four dimensions and find the representation wavefunction. We also compare with Wigner's solution for the CSR.

The plan of this dissertation is the following: chapter 1 is the introduction, chapter 2 reviews the CSR and extends it in any dimensions, chapter 3 discusses the supersymmetric extension of the CSRs, chapter 4 reviews the group contraction, chapter 5 discusses the contraction of the five dimensional regular massless representation, chapter 6 is the contraction of the massive representation in five dimensions, and finally chapter 7 is the conclusion.

CHAPTER 2

THE CONTINUOUS SPIN REPRESENTATIONS OF THE POINCARÉ GROUP

In this chapter, we introduce the continuous spin representations (CSRs) of the Poincaré group. For convenience, we express the generators in light-cone form [11, 12]. We start with the general form of the Poincaré algebra in any dimensions, and review and construct the continuous spin representation in four and five dimensions respectively. We also present the Casimirs for arbitrary dimension. More details can be found in Wigner [1] and Brink et al. [13].

2.1 Light-Cone Form of the Poincaré Algebra

The Poincaré group is the semi-direct product of the abelian translation group and homogeneous Lorentz group for rotations. The Poincaré generators satisfy the commutation relations,

$$[P^\mu, P^\nu] = 0, \quad (2.1)$$

$$[M^{\mu\nu}, P^\sigma] = i(\eta^{\mu\sigma} P^\nu - \eta^{\nu\sigma} P^\mu), \quad (2.2)$$

$$[M^{\mu\nu}, M^{\alpha\beta}] = i(\eta^{\mu\alpha} M^{\nu\beta} + \eta^{\alpha\nu} M^{\beta\mu} + \eta^{\nu\beta} M^{\mu\alpha} + \eta^{\beta\mu} M^{\alpha\nu}), \quad (2.3)$$

where $\eta^{\mu\nu} = (-1, 1, \dots, 1)$ and $\mu, \nu, \alpha, \beta = 0, 1, \dots, (d-1)$; and P^μ and $M^{\alpha\beta}$ are the generators for translation and rotation respectively. All representations of the Poincaré group have been classified by E. P. Wigner in a classic paper [1] in 1939. These are characterized by the Casimir operators which are the squared mass, M^2 , and the squares of the Pauli-Lubanski forms which in d space-time dimensions are defined as

$$W_{\mu_1 \dots \mu_n} = \frac{\epsilon_{\mu_1 \dots \mu_n \mu_{n+1} \dots \mu_d} P^{\mu_d} M^{\mu_{n+1} \mu_{n+2}} \dots M^{\mu_{d-2} \mu_{d-1}}}{\sqrt{n! 2^{\frac{d-n+1}{2}} \left(\frac{d-n-1}{2}\right)! \left(\frac{d-n-1}{2}\right)!}}, \quad (2.4)$$

where $n = 1, 3, \dots, (d-3)$ for d even and $n = 0, 2, \dots, (d-3)$ for d odd. Their squares are the Casimir operators of the light-cone little group. In the following, we rewrite the Poincaré group generators in light-cone language, originally introduced by Dirac [11] for spinless particles and later extended for arbitrary spin by Bacry and Chang [12], and characterize the representations by the Casimirs of the light-cone little group.

The light-cone variables are defined as

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^{d-1}), \quad p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^{d-1}),$$

and similarly for any other variables. In this formalism, the canonical commutation relations become

$$[x^\pm, p^\mp] = -i \quad \text{and} \quad [x^i, p^j] = i\delta^{ij},$$

where $i, j = 1, 2, \dots, (d-2)$ are the transverse directions. In light-cone form, the space-time translation generators become

$$P^- = \frac{p^i p^i + M^2}{2p^+}, \quad P^+ = p^+, \quad P^i = p^i,$$

where P^- , the light-cone Hamiltonian, is expressed in terms of transverse momenta by using the mass-shell condition. Since P^- , the conjugate variable to x^+ , is constrained by the mass-shell condition, we set $x^+ = 0$. The Lorentz generators can be grouped into two types: those which transform the transverse plane into itself are called "kinematic" and those which transform out of that plane are called "Hamiltonian" types respectively by Dirac. The kinematic generators are given by

$$M^{+i} = -x^i p^+, \quad M^{+-} = -x^- p^+, \quad M^{ij} = x^i p^j - x^j p^i + S^{ij}, \quad (2.5)$$

where S^{ij} obey the $SO(d-2)$ Lie algebra of the transverse little group

$$[S^{ij}, S^{kl}] = i(\delta^{ik} S^{jl} + \delta^{jl} S^{ik} - \delta^{il} S^{jk} - \delta^{jk} S^{il}). \quad (2.6)$$

The Hamiltonian-like boosts are

$$M^{-i} = x^- p^i - \frac{1}{2}\{x^i, P^-\} + \frac{1}{p^+}(T^i - p^j S^{ij}). \quad (2.7)$$

The vector, T^i , is called the light-cone translation vector. It transforms as a vector under the $SO(d-2)$ rotations,

$$[S^{ij}, T^k] = i(\delta^{ik} T^j - \delta^{jk} T^i), \quad (2.8)$$

and satisfies the following commutation relation,

$$[T^i, T^j] = iM^2 S^{ij}. \quad (2.9)$$

Eqs.(2.8-2.9) form the algebra of the little group. There are four ways to satisfy the algebra:

- $M \neq 0$: There are two types of representations in this case depending on whether M is real, positive or imaginary.

1. M real and positive: This is known as the massive representation. In this case, T^i/M and S^{ij} form the homogeneous little group $SO(d-1)$. In four dimensions, it is the $SO(3)$

little group and all massive elementary particles obey this representation. The states are labelled by the mass M and the spin j .

2. M imaginary: In this case, the little group is exactly the same as in the previous case. This is known as the tachyonic representation. No tachyonic particles have been observed in Nature. The tachyons are unphysical particles because these travel faster than the speed of light [2].

- $M = 0$: In this case, there are also two types depending on whether T^i 's are zero or not.

1. $T^i = 0$: This corresponds to the familiar massless representations which describe particles with a finite number of degrees of freedom. These are realized on states that satisfy

$$T^i | p^+, p^i; \{a_k\} \rangle = 0, \quad k = 1, \dots, \text{Rank of } SO(d-2).$$

where $\{a_k\}$ are the Dynkin labels of $SO(d-2)$ representations. In four dimensions, the little group is $SO(2)$ and its eigenvalue helicity along with the zero-mass uniquely determine the representation. The Pauli-Lubanski vector is light-like.

2. $T^i > 0$: In this case, T^i are the c-number components of a transverse vector. The vectors T^i and the rotation generators S^{ij} form the $E(d-2)$ inhomogeneous little group. The states are labelled by the length of the light-cone vector and the subgroup $SO(d-3)$ of the $SO(d-2)$ little group which leaves T^i invariant,

$$T^i | p^+, p^i; \xi^i, \{a_k\} \rangle = \xi^i | p^+, p^i; \xi^i, \{a_k\} \rangle, \quad k = 1, \dots, \text{Rank of } SO(d-3).$$

where $\{a_k\}$ are now the Dynkin labels of $SO(d-3)$ subgroup which we call the “short little group.” In four dimensions, there is no such group and the states are simply labelled by an space-like vector of constant magnitude $\xi^i, i = 1, 2$. These span two distinct representations, called “continuous spin representations” by Wigner [1] in his original work. The Pauli-Lubanski vector is space-like. The CSRs describe a massless state with an infinite number of integer-spaced helicities.

In the following sections, we review continuous spin representation in four dimensions, construct the state in detail in five dimensions, and finally end this chapter by introducing the general formula for the Casimirs of the little group in any dimensions.

2.2 Continuous Spin Representations

These representations are characterized by the zero mass and the finite length of the commuting light-cone translations T^i which together with S^{ij} form the $E(d-2)$ Euclidean little group in d -dimension. In the following, we first briefly review the properties of the CSR in four dimensions following Wigner's work (in light-cone formulation) and show how the infinite tower of helicity states is generated, and extend it into the five dimensions and calculate the Casimirs for higher dimensions.

2.2.1 Four Dimensions

In four dimensions, the little group is formed by the two-vector T^i and $SO(2)$ rotation. Together these satisfy the $E(2)$ algebra,

$$[T^i, T^j] = 0, \quad \text{and} \quad [S^{ij}, T^k] = i(\delta^{ik}T^j - \delta^{jk}T^i); \quad \text{for } i, j = 1, 2.$$

Since S^{12} and T^i do not commute, helicity is no longer a good quantum number; however $T^i T^i$ does commute. The two Casimirs that uniquely label the representation are the zero mass-squared and the square of the Pauli-Lubanski vector given by

$$W^2 = T^i T^i = \Xi^2,$$

where we used Eq.(2.4) for four dimensions and worked in a frame where $p^\dagger = 0$ for simplicity.

It follows that a finite boost creates an infinite number of integer-spaced helicities. To see the appearance of the infinite number of helicity states, we construct the light-cone raising and lowering operators,

$$T^\pm = T^1 \pm iT^2,$$

which satisfy the relations,

$$[S^{12}, T^\pm] = \pm T^\pm \quad \text{and} \quad [T^\pm, T^\mp] = 0.$$

It is easy to show that

$$S^{12}(T^\pm|>) = (m \pm 1)(T^\pm|>),$$

which clearly shows that the state $T^\pm|>$ has helicity $(m \pm 1)$. Therefore, an action of an infinitesimal boost,

$$M^\pm = M^{-1} \pm iM^{-2},$$

changes the helicity by one unit because of the following relation

$$M^\pm |p^+, p^i = 0; \Xi, \alpha> = \frac{T^\pm}{p^+} |p^+, p^i = 0; \Xi, \alpha>.$$

Therefore an action of a finite boost produces states with possible helicities ranging from minus to plus infinity in integer steps. This is an infinite dimensional representation. There are two types of representations: those with all integer (single-valued) and those with all half-odd integer (double-valued) helicities, known as bosonic and fermionic CSR respectively.

As we have stated in the introduction, the CSRs have no obvious physical applications, explicitly in local field theories, but the appearance of an infinite number of states may indicate a connection with non-local theories of extended objects. This motivates their study in more general contexts.

2.2.2 Five Dimensions

In five dimensions, the light-cone little group generator S^{ij} and T^i , for $i, j = 1, 2, 3$, satisfy the following $E(3)$ algebra,

$$[T^i, T^j] = 0, \quad \text{and} \quad [S^{ij}, T^k] = i(\delta^{ik}T^j - \delta^{jk}T^i).$$

Using Eq.(2.4), we get the following two Pauli-Lubanski forms,

$$\begin{aligned} 0\text{-form: } W &= \frac{1}{\sqrt{2}} \epsilon_{ijk} T^i S^{jk}, \\ 2\text{-form: } W_{ij} &= \frac{1}{\sqrt{2}} \epsilon_{ijk} T^k, \end{aligned} \quad (2.10)$$

where we used a frame $p^i = p^- = 0$. The square of the two-form

$$W_{ij}W^{ij} = T^k T^k = \Xi^2, \quad (2.11)$$

is one Casimir, identical to the four dimensional case. The other Casimir, the zero-form, can be thought of as the projection of the generator S^{jk} along T^i . The vector T^i then acts as a “quantization axis,” along which W assumes the values

$$\frac{W}{\sqrt{2}\Xi} = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \dots \quad (2.12)$$

It follows that there are two types of representations corresponding to each integer and half-odd integer values of $W/\sqrt{2}\Xi$. For each value of $W/\sqrt{2}\Xi$, there corresponds one infinite dimensional representation. Unlike four dimensions, there are infinite numbers of both bosonic and fermionic CSRs in higher dimensions. The states are no longer characterized by the light-cone little group but by its $SO(2)$ short little group orthogonal to T^i . Each CSR is labelled by the length of T^i and the eigenvalue m of the short little group which is the helicity.

It is straightforward to find the CSRs in terms of the eigenstates of $SO(3)$, the full little group. Let $|j, m\rangle$ be the eigenstate of the little group. These states are required to be eigenstates of $\vec{T}$, the $SO(2)$ rotations about it, and of the Casimir $\epsilon_{ijk}T^iS^{jk}$. Since T^i is a vector under $SO(3)$ rotation,

its action on each $|j, m\rangle$ yields a linear combination of $j = j, j \pm 1$ states. Therefore, the CSR eigenstates are infinite linear combinations of eigenstates of the full little group $SO(3)$. To construct a CSR state, let the light-cone vector be aligned along the third direction. This allows us to identify T^3 with the tensor operator T_0^1 which is the same as Y_0^1 in spherical basis. The action of T_0^1 on the state $|j, m\rangle$ yields

$$T_0^1 |j, m\rangle = \alpha_+^{(jm)} |j+1, m\rangle + \alpha_0^{(jm)} |j, m\rangle + \alpha_-^{(jm)} |j-1, m\rangle, \quad (2.13)$$

where the α 's are proportional to Clebsch-Gordan coefficients

$$\begin{aligned} \alpha_0^{(jm)} &= \Xi \sqrt{(j+m)(j-m+1)}, \quad -j < m < j \\ \alpha_{\pm}^{(jm)} &= \Xi \sqrt{(j'+m)(j'-m+1)}; \quad j' = j \pm 1, \quad -j' < m < j'. \end{aligned}$$

Let us define a state,

$$|F\rangle = \sum_{j=|m|}^{\infty} f_j^{(m)} |j, m\rangle.$$

This will be a CSR eigenstate if it is an eigenstate of both T^3 and the zero form Casimir,

$$\frac{1}{2} \epsilon_{ijk} T^i S^{jk} |F\rangle = \Xi m |F\rangle, \quad m = 0, \pm \frac{1}{2}, \pm 1, \dots$$

We find that, after using Eq.(2.13) and little algebra, the coefficients must satisfy the following recursion relations

$$\alpha_-^{(|m|+p, m)} f_{|m|+p}^{(m)} + \alpha_+^{(|m|+p-2, m)} f_{|m|+p-2}^{(m)} + (\alpha_0^{(|m|+p-1, m)} - \Xi) f_{|m|+p-1}^{(m)} = 0,$$

with $p = 2, 3, \dots$ and

$$\alpha_-^{(|m|+1, m)} f_{|m|+1}^{(m)} + (\alpha_0^{(|m|, m)} - \Xi) f_{|m|}^{(m)} = 0,$$

since $f_{|m|-1}^{(m)} = 0$. Clearly for each value of j , the $SO(3)$ angular momentum, there are two CSRs: bosonic and fermionic types corresponding to whether m is integer or half-odd integer, and there are an infinite number of CSRs of each type. The other CSR states can be obtained by acting the $SO(3)$ raising and/or lowering operator on $|F\rangle$.

2.2.3 Higher Dimensions

In higher dimensions, we just list the Casimirs and explain how the states are labelled, for more detail see Brink et al. [13]. Following the previous section, it is clear that the Casimirs of the short little group and the length-squared of the highest form uniquely label the representation.

From Eq.(2.4), the Pauli-Lubanski n -form for the light-cone little group $SO(d-2)$ can be written as

$$W_{i_1 \dots i_n} = \frac{\epsilon_{i_1 \dots i_n i_{n+1} \dots i_{d-2}} T^{i_{d-2}} S^{i_{n+1} i_{n+2}} \dots S^{i_{d-4} i_{d-3}}}{\sqrt{n! 2^{\frac{d-n-3}{2}} \left(\frac{d-n-3}{2}\right)!}}.$$

The Casimir operators are simplest to calculate in a frame where only $T^{i_{d-2}} \neq 0$. The square of the highest form is the squared length of the vector $\vec{T}$, a common feature for all dimensions. The Casimir operators of the short little groups for $n \geq 1$ are given by

- For little group $SO(4)$ and $SO(5)$:

$$W_{i_1 \dots i_n}^2 = \Xi^2 S_{\perp}^2; \quad 1 < d-n-3 \leq 3.$$

- For little group $SO(6)$ and $SO(7)$:

$$W_{i_1 \dots i_n}^2 = \Xi^2 \left\{ (S_{\perp}^2)^{(d-n-3)/2} - 2(S_{\perp}^{(d-n-3)}) \right\}; \quad 1 < d-n-2 \leq 5.$$

- For little group $SO(8)$ and $SO(9)$:

$$W_{i_1 \dots i_n}^2 = \Xi^2 \left\{ \begin{aligned} &(S_{\perp}^2)^{(d-n-3)/2} - 2(S_{\perp}^{(d-n-3)}) \quad ; \quad 1 < d-n-2 < 7. \\ &-(S_{\perp}^2)^3 + 2S_{\perp}^2 S_{\perp}^4 - 2S_{\perp}^6 \quad ; \quad d-n=9. \end{aligned} \right.$$

- In general for the little group $SO(d)$:

$$W_{i_1 \dots i_n}^2 = \Xi^2 \sum_{p=0}^k \left(A_{2p} S_{\perp}^{2p} S_{\perp}^{2(k-p)} + B_{2p} (S_{\perp}^2)^k + C_{2p} (S_{\perp}^2)^p S_{\perp}^{2(k-p)} \right),$$

where $k = \frac{1}{2}(d-n-3)$, and A_{2p} , B_{2p} and C_{2p} 's are numerical constants. In the above, $S_{\perp}^{ij}$ are the generators of $SO(d-3)$ subgroup of $SO(d-2)$ perpendicular to $T^{i_{d-2}}$ and

$$S_{\perp}^p \equiv S_{\perp}^{i_1 i_2} S_{\perp}^{i_2 i_3} \dots S_{\perp}^{i_{p-1} i_p} S_{\perp}^{i_p i_1}.$$

In odd dimensions only, the extra Casimir operator is provided by the Pauli-Lubanski zero-form

$$W = \frac{1}{\sqrt{2^{\frac{d-3}{2}} \left(\frac{d-3}{2}\right)!}} \epsilon_{ijk \dots mn} T^i S_{\perp}^{ij} \dots S_{\perp}^{mn}.$$

The CSR states are labelled by Ξ and the solid angle in $(d-3)$ dimensions Ω_{d-3} which give the length and direction of $\vec{T}$ respectively, as well as by $(a_1, \dots, a_r)$, the Dynkin labels of $SO(d-3)$,

$$|p^+, p^i; \Xi, \Omega_{d-3}; \{a_r\} \rangle,$$

where $\{a_r\}$ is the Dynkin label of the $SO(d-3)$ short little group with r being the rank. These differ from the usual massless representations in that they are characterized by a space-like vector, and contain an infinite number of states. In the next chapter, we extend the Poincaré group to include supersymmetry and find the supersymmetric CSR(SCSR).

CHAPTER 3 THE CSR IN SUPER-POINCARÉ ALGEBRA

The Super-Poincaré algebra includes the Poincaré algebra and the spinorial supercharge. In this chapter, we discuss how to construct the supersymmetric (SUSY) CSR. This chapter is based on Brink et al. [13]. A supermultiplet consists of an equal numbers of bosonic and fermionic states, and when a supercharge acts on these, they transform into each other. The supercharges are constructed using the Grassmann parameters. Therefore, to close the algebra, the lorentz generators must also be extended to include the Grassmann parameter dependent terms. Since the light-cone translations transform as a vector under the transverse rotations, it can be constructed without or with using the Grassmann parameters. The first one is of course the usual type of vector as in the previous chapter, whereas the second one is necessarily nilpotent because of the Grassmann parameter whose square is zero. In the following two sections, we investigate these two possibilities.

3.1 Super-Charges in Light-Cone Form

The generators of the Super-Poincaré algebra satisfy the commutation relations in Eqs.(2.1-2.3) and the following for the spinorial supercharge,

$$[Q_A, p^\mu] = 0, \quad (3.1)$$

$$[M^{\mu\nu}, Q_A] = -\frac{1}{2}(\Gamma^{\mu\nu} Q)_A, \quad (3.2)$$

$$\{Q_A, Q_B^\dagger\} = (\Gamma^\mu p_\mu \Gamma^0)_{AB}, \quad (3.3)$$

where A, B are spinor indices. In the above,

$$\Gamma^{\mu\nu} = -\frac{i}{2}[\Gamma^\mu, \Gamma^\nu],$$

where the Γ matrices satisfy the anticommutation relation,

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}; \quad \mu, \nu = 0, \dots, d.$$

The supercharges can be realized linearly using Grassmann variables and their derivatives [14] as

$$Q_A = \partial_A + \frac{1}{2}(\Gamma^\mu p_\mu \Gamma^0)_{AB} \bar{\theta}^B, \quad (3.4)$$

and their conjugates as

$$Q_C^\dagger = \bar{\partial}_C + \frac{1}{2}(\Gamma^\mu p_\mu \Gamma^0)_{DC} \theta^D, \quad (3.5)$$

where we used

$$\Gamma^0 \Gamma^\mu \Gamma^0 = \Gamma^\mu,$$

and $\partial_A = \frac{\partial}{\partial \theta^A}$, $\bar{\partial}_A = \frac{\partial}{\partial \bar{\theta}^A}$, and $\bar{\theta}_A$ is the complex conjugate of θ_A with $A, \dots, D$ running over the spinor indices. θ , $\bar{\theta}$, ∂ and $\bar{\partial}$ are anticommuting Grassmann parameters. Given the supercharges, it is always possible to define the covariant derivatives,

$$D_A = \partial_A - \frac{1}{2}(\Gamma^\mu p_\mu \Gamma^0)_{AB} \bar{\theta}^B,$$

and their hermitian conjugates. The covariant derivatives always anticommute with the supercharges and their conjugates,

$$\{Q_A, D_B\} = \{Q_A^\dagger, D_B\} = 0. \quad (3.6)$$

Acting on the wavefunction Φ , the above anticommutativity is equivalent to the following constraint,

$$D \Phi = 0, \quad (3.7)$$

which is a common constraint in any dimension. This indicates that the supercharges and hence all other generators are reducible. This constraint reduces the number of Grassmann parameters by a factor of two. There may be further reducibility depending on whether the Grassmann parameters are of Dirac type or Majorana type. These two constraints depend on the number of space-time dimensions. The irreducible form is obtained by imposing all constraints allowed. In d -dimension, a complex spinor have $2^{d/2}(2^{(d-1)/2})$ components for d even(odd). If p is the number of constraints imposed, the number of independent complex spinor is $2^{d/2-p}(2^{(d-1)/2-p})$ for d even(odd)¹. Let $a, b, c, \dots$ run over the irreducible components. Imposing all constraints, the irreducible forms of the kinematic supercharges become

$$Q_+^a = \frac{\partial}{\partial \theta^a} + \frac{1}{\sqrt{2}} p^+ \bar{\theta}^a, \quad Q_+^{†a} = \frac{\partial}{\partial \bar{\theta}^a} + \frac{1}{\sqrt{2}} p^+ \theta^a, \quad (3.8)$$

and the dynamic supercharges become

$$Q_-^a = f(p^a/p^+) Q_+^a, \quad Q_-^{†a} = f^*(p^a/p^+) Q_+^{†a}, \quad (3.9)$$

¹ The detailed analysis can be found in [15, 16].

where $f(p^a/p^+)$ and its complex conjugate are functions of the ratio $\frac{p^a}{p^+}$, determined by the constraint $D\Phi = 0$. The kinematic and dynamic components are determined by using the light-cone projector. The irreducible forms of the Lorentz generators can be written as

$$M^{ij} = x^i p^j - x^j p^i + \widehat{S}^{ij}, \quad (3.10)$$

$$M^{+-} = -x^- p^+ + S^{+-}, \quad (3.11)$$

$$M^{-i} = x^- p^i - \frac{1}{2}\{x^i, P^-\} + \frac{1}{p^+}(T^i - p^j \widehat{S}^{ij}) - \frac{p^i}{p^+} S^{+-}, \quad (3.12)$$

where

$$\widehat{S}^{ij} = S^{ij} + \left(\frac{1}{2} \theta^a (\gamma^{ij})_{ab} \theta^b + \text{c.c.} \right), \quad (3.13)$$

$$S^{+-} = \left(\frac{1}{2} \theta^a (\gamma^{+-})_{ab} \theta^b + \text{c.c.} \right), \quad (3.14)$$

with γ^{ij} and γ^{+-} being the reduced Dirac submatrices consistent with the constraints. M^{+i} remain the same as before. The indices i, j run over transverse space.

3.2 Non-nilpotent Light-cone Vector and the SUSY CSR

In this section, the light-cone vector is considered like an ordinary vector as in the Poincaré group. Following the discussion in the previous section, we write down the irreducible generators below and check various commutator whether the Super-Poincaré algebra closes or not.

3.2.1 Four Dimensions

In four dimensions, a complex spinor has four components. We can impose either chirality or Majorana condition and the covariant derivative condition, a common constraint for any dimensions, to reduce the number of independent component to one. Let θ be the Grassmann parameter. In Weyl representation, following Eq.(3.4) and imposing the chirality condition and $D\Phi = 0$, we can write the kinematic supercharge [13, 14] as

$$Q_+ = \frac{\partial}{\partial \bar{\theta}} + \frac{1}{\sqrt{2}} p^+ \bar{\theta}, \quad Q_+^\dagger = \frac{\partial}{\partial \theta} + \frac{1}{\sqrt{2}} p^+ \theta, \quad (3.15)$$

and the dynamic supercharges are given by

$$Q_- = \frac{p}{p^+} Q_+, \quad Q_-^\dagger = \frac{\bar{p}}{p^+} Q_+^\dagger, \quad (3.16)$$

where $p(\bar{p}) = \frac{1}{\sqrt{2}}(p^1 \pm ip^2)$. The generators M^{+-} , M^{12} and M^{-a} can be expressed in the following irreducible forms,

$$M^{+-} = -x^- p^+ - \frac{i}{2} \left(\theta \frac{\partial}{\partial \theta} + \bar{\theta} \frac{\partial}{\partial \bar{\theta}} \right), \quad (3.17)$$

$$M^{12} = x^1 p^2 - x^2 p^1 + S^{12} + \frac{1}{2} \left(\theta \frac{\partial}{\partial \theta} - \bar{\theta} \frac{\partial}{\partial \bar{\theta}} \right), \quad (3.18)$$

$$M^{-a} = x^- p^a - \frac{1}{2} \{x^a, P^-\} + \frac{1}{p^+} (T^a - p^b \hat{S}^{ab}) + i \frac{p^a}{2p^+} \left(\theta \frac{\partial}{\partial \theta} + \bar{\theta} \frac{\partial}{\partial \bar{\theta}} \right), \quad (3.19)$$

where $a, b = 1, 2$. The above generators and the supercharges have to satisfy the following commutation relations to close the algebra,

$$[M^{+-}, Q_{\pm}] = \pm \frac{i}{2} Q_{\pm}, \quad (3.20)$$

$$[M^{12}, Q_{\pm}] = \mp \frac{1}{2} Q_{\pm}, \quad (3.21)$$

$$[M^{+i}, M^{-j}] = i(\delta^{ij} M^{+-} - M^{ij}), \quad (3.22)$$

$$[M^{\pm a}, Q_{\pm}] = 0, \quad (3.23)$$

$$[M^{\pm 1}, Q_{\mp}] = -\frac{i}{\sqrt{2}} Q_{\pm}, \quad (3.24)$$

$$[M^{\pm 2}, Q_{\mp}] = -\frac{1}{\sqrt{2}} Q_{\pm}, \quad (3.25)$$

which are derived by applying the constraints to Eqs.(3.1-3.3) directly. After a long and tedious calculation, it can be shown that the irreducible generators satisfy these commutation relations. However, the last two relations, Eqs.(3.24-3.25), require that Q_+ to commute with T^a . An easier to see this is to choose a frame where $p^i = 0$. In this frame, the dynamic supercharge vanishes and these commutators become

$$[T^a, Q_+] = 0. \quad (3.26)$$

Since the light-cone vector commutes with the supercharge, we can implement supersymmetry on the continuous spin representations without having to change the supercharges. Let $|m\rangle$ be a state with helicity m . So its supersymmetric partner $Q_+|m\rangle$ has helicity $(m - 1/2)$. So an action of a finite boost on the states $|m\rangle$ and $|m - 1/2\rangle$ generates the CSR states $|m\rangle_{\text{CSR}}$ and $|m - 1/2\rangle_{\text{CSR}}$ respectively. So the CSR supermultiplet is

$$| \rangle_{\text{CSR}} = \begin{pmatrix} |m\rangle_{\text{CSR}} \\ |m - 1/2\rangle_{\text{CSR}} \end{pmatrix}, \quad (3.27)$$

which is like ordinary ($T^i = 0$) supermultiplet, but accompanied by an infinite tower of states.

3.2.2 Five Dimensions

In five dimensions, the spinors have four complex components which can be reduced to two by imposing the covariant derivative constraint, the only constraint allowed. In the representation,

$$\Gamma^0 = i\sigma^1 \otimes I, \quad \Gamma^i = \sigma^3 \otimes \sigma^i, \quad i = 1, 2, 3; \quad \text{and} \quad \gamma^4 = \sigma^2 \otimes I, \quad (3.28)$$

the light-cone projectors, $\mathcal{P}_\pm = -\frac{1}{2}\Gamma^\mp\Gamma^\pm$, split up Eq.(3.3) into the following,

$$\left\{ Q_{\pm a}, Q_{\pm b}^\dagger \right\} = \sqrt{2} p^\pm \delta_{ab} \quad \text{and} \quad \left\{ Q_{\pm a}, Q_{\mp b}^\dagger \right\} = \pm i (\vec{\sigma} \cdot \vec{p})_{ab}, \quad (3.29)$$

where $a = 1, 2$. The constraint, $D\Phi = 0$, gives the following relation,

$$\frac{\partial}{\partial \theta^{2+a}} = -i \left(\frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{2} p^+} \right)_{ab} \frac{\partial}{\partial \theta^b},$$

and similarly for their complex conjugates. This allows us to set $(\theta^3, \theta^4) = 0$. Substituting these, the supercharges in Eqs.(3.4-3.5) reduce to the irreducible forms

$$Q_+^a = \frac{\partial}{\partial \theta^a} + \frac{1}{\sqrt{2}} p^+ \vec{\theta}^a, \quad Q_+^{a\dagger} = \frac{\partial}{\partial \theta^a} + \frac{1}{\sqrt{2}} p^+ \theta^a, \quad (3.30)$$

with $a, b = 1, 2$, and the dynamic supercharges are now given by

$$Q_{-a} = -i \left(\frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{2} p^+} Q_+ \right)_a \quad \text{and hence} \quad Q_{-a}^\dagger = i \left(Q_+^\dagger \frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{2} p^+} \right)_a. \quad (3.31)$$

The internal part of the Lorentz generators S^{+-} and $\hat{S}^{ij}$ become

$$S^{+-} = -\frac{i}{2} \left(\theta \frac{\partial}{\partial \theta} + \vec{\theta} \frac{\partial}{\partial \vec{\theta}} \right), \quad \hat{S}^{ij} = S^{ij} + \frac{1}{2} \epsilon^{ijk} \left(\theta \sigma^k \frac{\partial}{\partial \theta} + c.c. \right), \quad (3.32)$$

where now $\theta = (\theta^1, \theta^2)$ and $\frac{\partial}{\partial \vec{\theta}} = \left(\frac{\partial}{\partial \theta^1}, \frac{\partial}{\partial \theta^2} \right)$. The remaining Lorentz generators remain unchanged. It is quite easy to see that the supercharges transform as $SO(3)$ spinors,

$$[M^{ij}, Q_{+a}] = -\frac{1}{2} \epsilon^{ijk} (\sigma^k Q_+)_{+a}. \quad (3.33)$$

These charges can be made to act on the CSRs, for which the relevant group is that of the short little group $SO(2)$ which leaves the light-cone translation vector invariant. Let us align T^4 along the third direction. This causes the supercharges to split up into two types according to their action on a helicity state,

$$[M^{12}, Q_{+1}] = -\frac{1}{2} Q_{+1}, \quad [M^{12}, Q_{+2}] = \frac{1}{2} Q_{+2}. \quad (3.34)$$

Clearly the supercharges Q_{+1} and $Q_{+2}^\dagger$ lower the helicity and the remaining two Q_{+2} and $Q_{+1}^\dagger$ raise it. In terms of the helicity m , the supermultiplet consists of the following states,

$$\begin{aligned} |m\rangle_{\text{CSR}} &\sim |m\rangle_{\text{CSR}} , \\ Q_{+1}|m\rangle_{\text{CSR}} \quad \text{and} \quad Q_{+2}^\dagger|m\rangle_{\text{CSR}} &\sim |m - \frac{1}{2}\rangle_{\text{CSR}} , \\ Q_{+1}Q_{+2}^\dagger|m\rangle_{\text{CSR}} &\sim |m - 1\rangle_{\text{CSR}} . \end{aligned}$$

It contains two bosonic and two fermionic CSRs, with the same structure as the ordinary ($\vec{T} = 0$) massless $N = 2$ supermultiplet in four dimensions. The important difference is that it contains not only the ordinary states but their copies under the boosts proportional to $\vec{T}$. This yields as usual an infinite number of $SO(3)$ polarization states. The action of supersymmetry is the same as in the normal case, but the CSR supermultiplets contain an infinite number of ordinary massless supermultiplets of ever-increasing spin. This construction has obvious generalization to any higher dimensions.

3.3 Nilpotent Light-Cone Translations and the CSR

In the previous section, we have shown that the CSR has supersymmetric extension. The light-cone translation vector was ordinary type, i.e., as in Poincaré algebra. In this section, we construct the light-cone vectors T^i using the Grassmann parameters. This construction will, of course, make the light-cone vector nilpotent, such that at least T^2 is nonzero but the higher orders are. If T^2 becomes zero this will not lead to any CSR. A finite boost will now terminate after the second order term resulting a finite number of helicity state. Since it is not possible to construct $SO(d)$ vector using only one set of Grassmann parameter, we have to introduce at least two sets of Grassmann parameters which automatically make T^2 nonzero, but the higher orders are zero. This can be thought of $N = 2$ supersymmetry. It should be noted that if we require higher degree of nilpotency, we have to introduce more Grassmann parameters and thereby will be able to consider higher N supersymmetry. In the following, we construct nilpotent light-cone translation in four and ten dimensions.

3.3.1 Four Dimensions

Since it is not possible to build a $SO(2)$ vector with one Grassmann parameter, we introduce two complex Grassmann variables θ_1 and θ_2 . We define the light-cone vectors as

$$T^1 = \frac{p^+}{\sqrt{2}}(Z\bar{\theta}_1\bar{\theta}_2 + \bar{Z}\theta_1\theta_2) , \quad (3.35)$$

$$T^2 = \frac{ip^+}{\sqrt{2}}(Z\bar{\theta}_1\bar{\theta}_2 - \bar{Z}\theta_1\theta_2) , \quad (3.36)$$

where Z is a complex c-number parameter, and p^+ is used to ensure its proper commutation relation with M^{+-} . The square of the Pauli-Lubanski vector is,

$$W^2 = 2|Z|^2(p^+)^2\theta_1\theta_2\bar{\theta}_1\bar{\theta}_2. \quad (3.37)$$

The kinematic supercharge and its conjugate are

$$Q_+^a = \frac{\partial}{\partial\theta_a} + \frac{1}{\sqrt{2}}p^+\bar{\theta}_a; \quad Q_+^{a\dagger} = \frac{\partial}{\partial\bar{\theta}_a} + \frac{1}{\sqrt{2}}p^+\theta_a, \quad (3.38)$$

where $a = 1, 2$, because of two Grassmann parameters. This is equivalent to $N = 2$ supersymmetry. However the light-cone translations no longer commute with these supercharges. To close the Super-Poincaré algebra, the dynamic supercharges must be altered to the new form

$$Q_-^a = \frac{p}{p^+} Q_+^a - i\bar{Z}\epsilon^{ab}\theta_b, \quad Q_-^{a\dagger} = \frac{\bar{p}}{p^+} Q_+^{a\dagger} + iZ\epsilon^{ab}\bar{\theta}_b, \quad (3.39)$$

which ensure the commutation relations of Q_+^a with the boosts M^{-a} . These supercharges also transform correctly under transverse rotation S^{12} which now reads,

$$\hat{S}^{12} \equiv S^{12} + \frac{1}{2}(\theta^a\partial_a - \bar{\theta}^a\bar{\partial}_a). \quad (3.40)$$

Let us compute the anticommutator between the supercharges. We find that,

$$\{Q_+^a, Q_-^b\} = i\epsilon^{ab}\bar{Z}, \quad \{Q_+^{a\dagger}, Q_-^{b\dagger}\} = -i\epsilon^{ab}Z. \quad (3.41)$$

These non-zero anticommutator together with the Super-Poincaré algebra (anti)commutators form the supersymmetry algebra with central charge, even though the representation is massless. It is well-known that the supersymmetry with central charge leads to negative or zero norm states for massless representation [16], it only makes physical sense for massive representation.

Even though by constructing the nilpotent light-cone translation vector, we remedied the infinite vacuum heat capacity problem which was Wigner's argument against the existence of the CSR, but at the expense of negative or zero norm states.

3.3.2 Ten Dimensions

Following previous section, the ten dimensional case is straightforward. The light-cone little group is $SO(8)$ which has the magic triality property. In ten dimensions, we can impose both Majorana and chirality conditions, in addition to the $D\Phi = 0$ constraint. These reduce the number of independent real spinor components to eight. As in four dimensions, we use two sets of real

Grassmann parameters to form $SO(8)$ vector. In the representation,

$$\Gamma^0 = i\sigma^1 \otimes I, \quad \Gamma^i = \sigma^3 \otimes \gamma^i, \quad (3.42)$$

where $i = 1, \dots, 9$ and γ^i 's are 8×8 gamma matrices, the supercharges and the light-cone vectors can be written as

$$Q_+^{(a)} = \frac{\partial}{\partial \theta_a} + \frac{1}{\sqrt{2}} p^+ \theta_a; \quad a = 1, 2. \quad \text{and} \quad T^i = ip^+ Z \theta_1 \gamma^i \theta_2, \quad (3.43)$$

where Z is now real. These transform as

$$Q_+^{(1)} \sim 8_s, \quad Q_+^{(2)} \sim 8_c \quad \text{and} \quad T^i \sim 8_v,$$

under the $SO(8)$ transverse rotations given by

$$\hat{S}^{ij} = S^{ij} - \frac{i}{2} \left(\theta_1 \gamma^{ij} \frac{\partial}{\partial \theta_1} + \theta_2 \gamma^{ij} \frac{\partial}{\partial \theta_2} \right), \quad (3.44)$$

where γ^{ij} is the reduced tensor. The generator S^{+-} and the dynamic boost reduce to the following forms,

$$S^{+-} = -\frac{i}{2} \left(\theta_1 \frac{\partial}{\partial \theta_1} + \theta_2 \frac{\partial}{\partial \theta_2} \right) \quad \text{and} \quad S^{-i} = i Z \theta_1 \tilde{\gamma}^i \theta_2 - \frac{p^j}{p^+} \hat{S}^{ij} - \frac{p^i}{p^+} S^{+-}.$$

To satisfy the algebra, the dynamic supercharges must change to the new form,

$$Q_-^{(a)} = -\frac{ip^i}{\sqrt{2}p^+} \gamma^i Q_+^{(a)} - i\epsilon^{ab} \sqrt{2} Z \theta, \quad a, b = 1, 2. \quad (3.45)$$

where $\epsilon^{12} = 1 = -\epsilon^{21}$. The anticommutators between the kinematic and dynamics supercharges are no longer zero,

$$\{Q_+^{(a)}, Q_-^{(b)}\} = i\epsilon^{ab} \sqrt{2} Z, \quad (3.46)$$

indicating the $N = 2$ supersymmetry with central charge. Even though our construction leads to supersymmetry but the representations necessarily contain negative and zero-norm states. Although it is interesting to note that central charges occur naturally whenever the light-cone translations are built out of the Grassmann variables, the representations contain negative and zero-norm states; at best they could be used as ghost compensators of some unknown theory.

This construction does not seem to generalize to odd dimensions. To see it, consider eleven dimensions with $\vec{T}$ quadratic in the Grassmann numbers. There a quadratic product of a Grassmann spinor transforms as 2- and 3-forms, so that to make a vector we need some c-number tensors, either a one or two-form, but the commutation with the supercharge does not have the right form.

CHAPTER 4 INÖNÜ-WIGNER GROUP CONTRACTION

In this chapter, we review the group contraction first introduced by İnönü and Wigner [9], and also independently by Segal [10]. Later Saletan [17] discussed it for arbitrary homogeneous Lie groups in more general context. Here we restrict to the IW type contraction. We start with a brief review of the IW's original approach to group contraction, and then we show that the same result can be reproduced by imposing periodic boundary condition to the homogeneous $SO(3)$ group and taking two singular limits. The difference between these two approaches is the interpretation of the contraction parameter.

4.1 IW Group Contraction

Inönü and Wigner [9] used homogeneous $SO(3)$ group to introduce the group contraction. They showed explicitly how to obtain the Euclidean group $E(2)$ by contracting the $SO(3)$ group under singular limit. They applied the contraction to the algebra generators, identifying those which are well defined under contraction, and derived the two-dimensional wavefunction on which the Euclidean translation vectors act, the Bessel functions J_n .

Let S^i ($i = 1, 2, 3$) be the generator of the $SO(3)$ group which satisfy the angular momentum algebra¹,

$$[S_i, S_j] = i \epsilon_{ijk} S_k.$$

The states are labelled by the eigenvalue of the Casimirs S^2 and S_3 ,

$$S_i S_i = s(s+1), \quad S_3 = m.$$

Since $-s \leq m \leq s$, this spans a $(2s+1)$ -dimensional representation. The representation wavefunction is of course the spherical harmonics.

Let us define another set of generators as

$$S^{i'} = \lim_{\epsilon \rightarrow 0} \epsilon S^i,$$

¹ S^i is related to the generators S^{ij} by Poincaré duality, i.e., $S^i = \frac{1}{2} \epsilon^{ijk} S^{jk}$.

where ϵ is an arbitrary parameter, called the contraction parameter. For fixed m , $S^{3'} = 0$, and the remaining two generators and S^3 satisfy the following algebra,

$$[S_3, S^{a'}] = i\epsilon_{ab} S^{b'}, \quad [S^{a'}, S^{b'}] = 0, \quad (4.1)$$

where $a, b = 1, 2$. These commutation relations form the Euclidean algebra only if the Casimir is also nonzero. To find the Casimir, we multiply the $SO(3)$ Casimir by the contraction parameter,

$$(\epsilon S^3)^2 + (\epsilon S^a)^2 = \epsilon^2 s(s+1),$$

and taking the limit $\epsilon \rightarrow 0$, while keeping m fixed, yields the following,

$$(S^{a'})^2 = \lim_{\epsilon \rightarrow 0} \epsilon^2 s(s+1) = \begin{cases} 0 & ; \text{ if } s = \text{fixed.} \\ \Xi^2 & ; \text{ if } s \rightarrow \infty. \end{cases} \quad (4.2)$$

where $\Xi = \epsilon s$ is nonzero finite as $\epsilon \rightarrow 0$ and $s \rightarrow \infty$. In the first case, we obtain homogeneous $SO(2)$ group with S^3 being the Casimir operator, whereas the second case yields the Euclidean group under the double limit. Since s sets the range of m , and $s \rightarrow \infty$, m can now have infinite range. Thus it forms an infinite dimensional representation. This is the most essential feature of the contraction process. The Casimir eigenvalue of the $E(2)$ group now reads,

$$S^{a'} S^{a'} = \Xi^2. \quad (4.3)$$

Since $S^{a'}$ s are like momenta, rather than deriving from the $SO(3)$ generator, IW identified it with the differential operator for translation,

$$S^{a'} = -i \frac{\partial}{\partial x^a}, \quad a = 1, 2. \quad (4.4)$$

Therefore, Eq.(4.3) reduces to the Laplace wave equation in two dimensions. In cylindrical coordinates, the solution of the wave equation is the well-known Bessel function,

$$\Phi(\rho, \theta) = \sum_m N_m J_m(\Xi \rho) e^{im\theta},$$

where $\rho = \sqrt{(x^1)^2 + (x^2)^2}$. This is the representation wavefunction of the $E(2)$ algebra. In their paper, they also derived the following identity,

$$\lim_{l \rightarrow \infty} \left(\frac{(l-m)!}{(l+m)!} \right)^{1/2} P_l^m(\cos(\Xi \rho/l)) = J_m(\Xi \rho),$$

which justifies that the contracted algebra is well defined, in particular the identification of the contracted generator $S^{a'}$ with the differential operator for translation in Eq.(4.4). In their original paper, İnönü and Wigner applied their method to the Poincaré group, using the inverse speed of light as a contraction parameter, and find it to contract to the non-relativistic Galilei group *only if the starting point is the tachyonic representation!*

4.2 Contraction of $SO(3)$ group with periodic Boundary Condition

In this section, we apply the periodic boundary condition to the $SO(3)$ algebra by putting one direction on to a circle of radius R . The contraction parameter is now the inverse radius. We show that under contraction, the representation wavefunction is the same as the previous section. In the following, we retrace IW's steps, stressing the geometrical picture of the contraction procedure. The IW contraction of $SO(3)$ to $E(2)$ amounts to the study of a dynamical system with $SO(3)$ symmetry restrained to a space whose boundary condition breaks that symmetry. We switch to cylindrical coordinates, and seek solutions which are periodic in z ,

$$z \rightarrow z + 2\pi R, \quad (4.5)$$

$$\Phi(\rho, \theta, z) = \sum_n \Phi_n(\rho, \theta) e^{inz/R}, \quad (4.6)$$

where n is the mode number. Let j^2 be the eigenvalue of the Casimir $(S^{ij})^2$ when it acts on this wavefunction. Note that in spherical basis the Casimir eigenvalue is $j(j+1)$. For very large spin $j^2 \approx j(j+1)$, and eventually we will take it to infinity. So we no longer distinguish between j^2 and $j(j+1)$. Therefore, we write,

$$S^{ij} S^{ij} = j^2, \quad (4.7)$$

where $i, j = 1, 2, 3$. In the following, we use the inverse radius, $1/R$, as the contraction parameter.

The angular momentum operators are given by

$$S^{12} = -i \frac{\partial}{\partial \theta}, \quad (4.8)$$

$$S^{23} = -i \left(-z \sin \theta \frac{\partial}{\partial \rho} + \rho \sin \theta \frac{\partial}{\partial z} - \frac{z \cos \theta}{\rho} \frac{\partial}{\partial \theta} \right), \quad (4.9)$$

$$S^{31} = -i \left(z \cos \theta \frac{\partial}{\partial \rho} - \rho \cos \theta \frac{\partial}{\partial z} - \frac{z \sin \theta}{\rho} \frac{\partial}{\partial \theta} \right). \quad (4.10)$$

Acting on the wave function, the factor $\frac{\partial}{\partial z}$ can be replaced by $\frac{in}{R}$. Substituting this, the square of the generators become

$$(S^{12})^2 = -\frac{\partial^2}{\partial \theta^2}, \quad (4.11)$$

$$(S^{23})^2 = -z^2 \left[\sin^2 \theta \frac{\partial^2}{\partial \rho^2} + \frac{\cos^2 \theta}{\rho} \frac{\partial}{\partial \rho} + \frac{2 \sin \theta \cos \theta}{\rho} \frac{\partial^2}{\partial \theta \partial \rho} - \frac{\rho \sin^2 \theta}{z^2} \frac{\partial}{\partial \rho} \right. \\ \left. - \frac{2 \rho \sin^2 \theta}{z} \frac{in}{R} \frac{\partial}{\partial \rho} - \frac{2 \sin \theta \cos \theta}{\rho^2} \frac{\partial}{\partial \theta} + \frac{\cos^2 \theta}{\rho^2} \frac{\partial^2}{\partial \theta^2} - \frac{\sin \theta \cos \theta}{z^2} \frac{\partial}{\partial \theta} \right. \\ \left. - \frac{2 \sin \theta \cos \theta}{z} \frac{in}{R} \frac{\partial}{\partial \theta} - \frac{1}{z} \frac{in}{R} - \frac{\rho^2 \sin^2 \theta}{z^2} \frac{n^2}{R^2} \right], \quad (4.12)$$

$$(S^{31})^2 = -z^2 \left[\cos^2 \theta \frac{\partial^2}{\partial \rho^2} + \frac{\sin^2 \theta}{\rho} \frac{\partial}{\partial \rho} - \frac{2 \sin \theta \cos \theta}{\rho} \frac{\partial^2}{\partial \theta \partial \rho} - \frac{\rho \cos^2 \theta}{z^2} \frac{\partial}{\partial \rho} \right. \\ \left. - \frac{2 \rho \cos^2 \theta}{z} \frac{in}{R} \frac{\partial}{\partial \rho} + \frac{2 \sin \theta \cos \theta}{\rho^2} \frac{\partial}{\partial \theta} + \frac{\sin^2 \theta}{\rho^2} \frac{\partial^2}{\partial \theta^2} + \frac{\sin \theta \cos \theta}{z^2} \frac{\partial}{\partial \theta} \right. \\ \left. + \frac{2 \sin \theta \cos \theta}{z} \frac{in}{R} \frac{\partial}{\partial \theta} - \frac{1}{z} \frac{in}{R} - \frac{\rho^2 \cos^2 \theta}{z^2} \frac{n^2}{R^2} \right]. \quad (4.13)$$

When R is very large, all terms proportional to inverse R can be dropped². Using Eqs.(4.11-4.13) into Eq.(4.7) and dropping the terms proportional to $1/R$, we obtain the following differential equation,

$$-z^2 \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \left(\frac{1}{\rho^2} + \frac{1}{z^2} \right) \frac{\partial^2}{\partial \theta^2} \right) = j^2, \quad (4.14)$$

which acts on the wavefunction of the form

$$\Phi(\rho, \theta) = \mathcal{R}(\rho)\Theta(\theta). \quad (4.15)$$

Since there is no mode dependence, we have dropped the subscript n from the wavefunction. For the above form of the wavefunction, the angular part decouples from the radial part and can be written as

$$\frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} \equiv -m^2, \quad (4.16)$$

where m , the azimuthal quantum number, is an integer or half-odd integer depending on whether the wave function is single-valued or double-valued. Substituting this and rearranging, the differential equation Eq.(4.14) reduces to

$$\rho^2 \frac{d^2}{d\rho^2} + \rho \frac{d}{d\rho} + \left(\frac{j^2 - m^2}{z^2} \right) \rho^2 - m^2 = 0, \quad (4.17)$$

² The complete solution with finite R is the Whittaker function. See Appendix B.

which is the Bessel differential equation. The regular solution is

$$\Phi(\rho, \theta) = \sum_m N_m J_m(q\rho) e^{im\theta}, \quad (4.18)$$

where the scale factor is given by $q = \left(\frac{z^2 - m^2}{z^2}\right)$. This is the representation wavefunction for very large radius R .

Let us now define,

$$S^{a'} = \lim_{R \rightarrow \infty} \epsilon_{ab} \frac{S^{bs}}{R}, \quad (4.19)$$

while keeping m , eigenvalue of S^3 , fixed. Using Eqs.(4.9-4.10), we get,

$$S^{1'} = i \frac{z}{R} \left(-\sin \theta \frac{\partial}{\partial \rho} - \frac{\cos \theta}{\rho} \frac{\partial}{\partial \theta} \right), \quad (4.20)$$

$$S^{2'} = -i \frac{z}{R} \left(\cos \theta \frac{\partial}{\partial \rho} - \frac{\sin \theta}{\rho} \frac{\partial}{\partial \theta} \right). \quad (4.21)$$

It can easily be checked that the $S^{a'}$'s, $a = 1, 2$, commute with each other and that they transform as a 2-vector under S^{12} , forming a representation of the $E(2)$ algebra as in Eq.(4.1).

Let Ξ^2 be the eigenvalue of the Casimir operator for infinite radius limit. Therefore, the Casimir operator

$$(S^{1'})^2 + (S^{2'})^2 = \Xi^2, \quad (4.22)$$

reduces to the following differential equation,

$$-\frac{z^2}{R^2} \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2}{\partial \theta^2} \right) = \Xi^2, \quad (4.23)$$

where we used $\lim_{R \rightarrow \infty} \frac{S^3}{R} = 0$ and Eqs.(4.20-4.21). For the wavefunction of the form as in Eq.(4.15) and the angular dependence as in Eq.(4.16), the above differential equation becomes (after some rearrangement)

$$\rho^2 \frac{d^2}{d\rho^2} + \rho \frac{d}{d\rho} + \frac{\Xi^2}{(z^2/R^2)} \rho^2 - m^2 = 0, \quad (4.24)$$

whose regular solution is the well-known Bessel function,

$$\Phi(\rho, \theta) = \sum_m N_m J_m(\bar{q}\rho) e^{im\theta}, \quad (4.25)$$

where the scale factor is defined as $\bar{q} = \frac{\Xi^2}{(z^2/R^2)}$. Since the solutions in Eq.(4.18) and in Eq.(4.25) have to be identical for infinite radius limit, this implies that the scale factors in Eqs.(4.17,4.24) must also be identical. Therefore, equating these two scale factors, we obtain a relation between the

Casimir eigenvalues of the $SO(3)$ and $E(2)$ groups,

$$\Xi^2 = \frac{j^2}{R^2}, \quad (4.26)$$

where now both j and R tend to infinity, but their ratio Ξ remains fixed and nonzero. This solution is of course the same as that of İnönü and Wigner. The generalization of the above construction to any dimensions is also quite straightforward. In general, for d dimensions, compactifying one direction gives the following relation

$$\Xi_{d-1}^2 = \frac{j_d^2}{R^2}, \quad (4.27)$$

where j_d^2 is the quadratic Casimir of the homogeneous group $SO(d)$ and Ξ_{d-1}^2 is the length squared of the $E(d-1)$ group with R being the radius of the circle.

CHAPTER 5

CONTRACTION OF THE FIVE DIMENSIONAL MASSLESS REPRESENTATIONS

In this chapter, we discuss how the contraction of the five dimensional regular massless representation yields the CSR in four dimensions. We apply the techniques developed in the previous chapter. Since the Poincaré group has an orbital and an internal part, we need to address how the contraction process effects these two. For the orbital rotation part, we show explicitly how the five-momentum and the rotation generators involving the compactified direction combine to produce the four-momentum. For the internal part, the scenario is the same as the IW group contraction process. We end this chapter by constructing the CSR in four-dimension and the representation wavefunction of the Poincaré group.

5.1 KK Reduction of the Poincaré Algebra

Kaluza and Klein [18, 19] reduced one dimension by putting it on to a circle of finite radius. We apply it to the five-dimensional Poincaré algebra, by putting the third direction on to a circle of radius R ,

$$x^3 = x^3 + 2\pi R .$$

The generators act on functions of the form

$$\Phi(x) = \sum_n \Phi_n(x^\mu) e^{inz^3/R} , \quad (5.1)$$

where $\mu = +, -, 1, 2$ and n is the mode number. The momentum along the third direction is quantized,

$$p^3 = \frac{n}{R} . \quad (5.2)$$

This changes the light-cone Hamiltonian and two transverse boosts in the following way,

$$P^- = \frac{p^a p^a + (p^3)^2 + M^2}{2p^+} \equiv \frac{p^a p^a + M_n^2}{2p^+} , \quad (5.3)$$

$$M^{-a} = x^- p^a - \frac{1}{2} \{x^a, P^-\} + \frac{1}{p^+} \left(T^a + \frac{n}{R} S^{3a} - p^b S^{ab} \right) , \quad (5.4)$$

where $a, b = 1, 2$ and M_n is the KK mass. The single state of mass M in five dimensions generates an equally spaced tower of states of mass M_n . Looking at M^{-a} we identify the light-cone translation

vector,

$$\hat{T}^a = T^a + \frac{n}{R} S^{3a}, \quad (5.5)$$

which satisfy the algebra

$$[S^{ab}, \hat{T}^c] = i(\delta^{ac}\hat{T}^b - \delta^{bc}\hat{T}^a), \quad (5.6)$$

$$[\hat{T}^a, \hat{T}^b] = iM_n^2 S^{ab}. \quad (5.7)$$

The generators S^{12} and $\frac{\hat{p}^a}{M_n}$ form the $SO(3)$ light-cone little group in four dimensions. Analyzing the mass term, we find that M_n increases with the mode number and gives the well-known infinite KK tower of masses starting at $M_0 = M$. These are the general expression for the KK reduction of the five dimensional representations.

For the KK reduction of the regular massless representation in five dimensions, we simply substitute $M = 0$ and $T^i = 0$, $i = 1, 2, 3$. The resulting Light-cone Hamiltonian, the transverse boosts and the light-cone translations become

$$P^- = \frac{p^a p^a + (p^3)^2}{2p^+} \equiv \frac{p^a p^a + M_n^2}{2p^+}, \quad (5.8)$$

$$M^{-a} = x^- p^a - \frac{1}{2}\{x^a, P^-\} + \frac{1}{p^+} \left(\frac{n}{R} S^{3a} - p^b S^{ab} \right), \quad (5.9)$$

$$\hat{T}^a = \frac{n}{R} S^{3a} \quad (5.10)$$

where now $M_n = \frac{n}{R}$. The little group is formed by the generators S^{12} and $\hat{T}^a$ and they satisfy the algebra in Eqs.(5.6-5.7).

The remaining Lorentz generators of the four-dimensional group, M^{+-} , M^{12} and M^{+a} for $a = 1, 2$ are not changed. Note that when evaluated at $x^3 = 2\pi R$, the generators that rotate into the third direction become (for large R) like the momenta

$$\begin{aligned} \frac{M^{-3}}{R} &= \frac{n}{R^2} x^- - 2\pi p^- - \frac{p^a}{p^+} \left(\frac{S^{3a}}{R} \right), \\ \frac{M^{3i}}{R} &= 2\pi p^i - \frac{n x^i}{R^2} + \frac{1}{R} S^{3i}, \\ \frac{M^{+3}}{R} &= -2\pi p^+. \end{aligned}$$

In the following two sections, we show in details how the contraction of the orbital and internal parts lead to the CSR in four dimensions.

5.2 Contraction of the Orbital Rotations

At the group level, the contraction of the orbital rotations results in shifting the parameters for translations. This has been studied in detail by E. Wigner, Y. Kim and others in [20, 21]. We

consider their construction for five dimensions with periodic boundary condition and contract it to four dimensions.

In five dimensions, there are five translational and ten rotational parameters. We choose to use usual space-time coordinates rather than light-cone coordinates (only for this section),

$$x^\mu = (t, x^1, x^2, x^3, x^4) \equiv (x^+, x^-, x^i), \quad i = 1, 2, 3.$$

We want to contract with respect to the subgroup which leaves the third direction invariant. Let us consider the boost between t and x^3 axes. Using $x^3 = 2\pi R$, the transformation equation can be written as

$$\begin{pmatrix} 1 \\ t' \\ 1 \end{pmatrix} = \begin{pmatrix} \cosh \beta & \frac{\sinh \beta}{2\pi R} & \frac{a_3}{2\pi R} \\ 2\pi R \sinh \beta & \cosh \beta & t_0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ t \\ 1 \end{pmatrix}, \quad (5.11)$$

where $\sinh \beta = v$ and $\cosh \beta = \sqrt{1+v^2}$, with v being the velocity along the third direction. Since there is no velocity along the compactified direction, we need to take the limit $v \rightarrow 0$ as $R \rightarrow \infty$ such that Rv remains fixed. This gives

$$t' = t + a_t + t_0, \quad (5.12)$$

where $a_t = 2\pi Rv$ is a constant and t_0 is an arbitrary initial parameter. That is, the parameter for translation along time is shifted by a_t as a result of contraction of the $t - x^3$ rotation matrix.

Let us now consider the rotation between x^i and x^3 axes for $i = 1, 2, 4$. The transformation equation for rotation can be written as

$$\begin{pmatrix} x^{i'} \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos \phi^i & 2\pi R \sin \phi^i & a_i \\ -\frac{\sin \phi^i}{2\pi R} & \cos \phi^i & \frac{a_3}{2\pi R} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x^i \\ 1 \\ 1 \end{pmatrix}, \quad (5.13)$$

where ϕ^i is the rotation angle. Since after the compactification, there is no rotation on the $i - 3$ plane, we need to take the limit $\phi^i \rightarrow 0$ as $R \rightarrow \infty$ such that $R\phi^i$ remains constant. This gives

$$x^{i'} = x^i + a_{x^i} + a_i, \quad (5.14)$$

where $a_{x^i} = 2\pi R\phi^i$ is a constant and a_i is an arbitrary initial parameter. From the above, we see that the rotations between the invariant direction and other directions of the original group become the translations of the contracted group which is a generic feature. The generators of the contracted

group can be read off using new shifted parameters. As for example, consider the rotation on the $3-1$ plane. The product of the group elements for translation along first direction and the rotation on $3-1$ plane is

$$e^{-ia_1 p^1} e^{-i\omega_{31} L^{31}} = e^{-ia_1 p^1} e^{-i2\pi(R\phi^1)\frac{L^{31}}{2\pi R}} \xrightarrow{R \rightarrow \infty} e^{-i(a_1 + a_{x^1})p^1},$$

which is again a translation along the first direction, but with a new parameter $(a_1 + a_{x^1})$. Therefore, using standard method we can write down the differential form for p^1 with $(a_1 + a_{x^1})$ being the translational parameter along the first direction and similarly for any other directions.

The light-cone forms of the above results are straightforward. We find,

$$x'^{\pm} = x^{\pm} + a^{\pm} + a_0^{\pm} \quad \text{and} \quad x'^i = x^i + a_i + a_{x^i}; \quad i = 1, 2 \quad (5.15)$$

where $a^{\pm} = \frac{1}{\sqrt{2}}(a_t \pm a_{x^4})$ and $a_0^{\pm} = \frac{1}{\sqrt{2}}(t_0 \pm a_4)$. Under boost along the forth direction, $x^{\pm}$ transform as

$$x^{\pm'} = e^{\pm\lambda} x^{\pm}, \quad (5.16)$$

where λ is the rapidity parameter and is given by

$$e^{\lambda} = \sqrt{\frac{1+v_4}{1-v_4}},$$

with v_4 being the velocity along the forth direction. The transverse directions remain unchanged. Therefore, in the boosted frame, the orbital rotation generators after contraction can be written as

$$\begin{aligned} L^{+-} &= -x^- p^+, & L^{+i} &= -x^i p^+, \\ L^{12} &= x^1 p^2 - x^2 p^1, & L^{-i} &= x^- p^i - \frac{1}{2}\{x^i, p^-\}, \end{aligned}$$

where $i = 1, 2$ and $x^+ = 0$ is used because of mass-shell condition. This is exactly the same form of the Poincaré algebra in four dimensions for spinless particles Dirac obtained while introducing the front form dynamics [11].

5.3 Contraction of the Internal Part and the CSR

In this section, we apply contraction to the internal part. The generators $\hat{T}^a$ and S^{12} satisfy the algebra (for $R \rightarrow \infty$),

$$[S^{ab}, \hat{T}^c] = i(\delta^{ac}\hat{T}^b - \delta^{bc}\hat{T}^a) \quad \text{and} \quad [\hat{T}^a, \hat{T}^b] = 0, \quad (5.17)$$

where $\hat{T}^a = \frac{1}{R}S^{3a}$ and $a, b, c = 1, 2$. Let the internal rotations are parameterized by three pairs of canonical coordinates and their conjugate momenta which satisfy the following commutation

relations,

$$[\eta^i, \pi^j] = i\delta^{ij}, \quad i, j = 1, 2, 3 = (a, 3).$$

Unlike the orbital case, we do not put the third coordinate η^3 on to a circle. Instead we assume η^3 is very large and so its conjugate momentum π^3 is very small. As $R \rightarrow \infty$, η^3 also goes to infinity such that $\frac{\eta^3}{R}$ is well defined. Therefore, in the infinite radius limit, the terms proportional to π^3 can be ignored, and the terms proportional to η^3 becomes a function of $y \sim \frac{\eta^3}{R}$. Note that being a ratio of the internal and external coordinates, the parameter y bears the signature of the extra dimension. This is the difference between the contraction process introduced in Sec.4.2 and the internal part in this section. Therefore, in the contraction limit, we can write,

$$\hat{T}^a = y\pi^a, \quad (5.18)$$

where $y = \frac{n\eta^3}{R}$. The light-cone vector differs by a factor of y from the Euclidean vector $S^{a'}$ defined by Wigner (see Sec.4.1). We now follow the same procedure as in Sec.4.1. The Casimir of the $SO(3)$ little group is,

$$(S^{12})^2 + \left(\frac{\hat{T}^a}{M_n}\right)^2 = j(j+1). \quad (5.19)$$

Divide by R^2 and taking $R \rightarrow \infty$ while keeping m fixed, we find,

$$\hat{T}^a \hat{T}^a = \frac{n^2 j^2}{R^2} = \begin{cases} 0 & ; \text{ if } n = 0. \\ 0 & ; \text{ if } n \neq 0, j = \text{fixed}, R \rightarrow \infty. \\ \Xi^2 & ; \text{ if } n \neq 0, j, R \rightarrow \infty, \frac{nj}{R} \equiv \Xi = \text{fixed}. \end{cases} \quad (5.20)$$

The first two clearly belong to the regular massless representation, and the last one is the CSR in four dimensions. Notice that for zero mode, the representation is always regular massless. Clearly only the double limit yields the CSR as expected. Combining Eqs.(5.18,5.20), we find,

$$(\hat{T}^a)^2 = y^2(\pi^a)^2 = \Xi^2, \quad (5.21)$$

which in cylindrical coordinates gives the differential equation,

$$\rho^2 \frac{d^2}{d\rho^2} + \rho \frac{d}{d\rho} + \left(\frac{\Xi^2 \rho^2}{y^2} - m^2\right) = 0. \quad (5.22)$$

The regular solution is the well-known Bessel function,

$$\Phi(\rho, \theta, y) = \sum_m N_m J_m(\Xi \rho / y) e^{im\theta}. \quad (5.23)$$

In their paper [9], IW obtained the following solution,

$$\Phi_{IW}(\rho, \theta) = \sum_m A_m e^{im\theta} J_m(\Xi\rho) . \quad (5.24)$$

At $y = 1$ these two solutions become identical. Therefore the complete Poincaré wavefunction can be written as

$$\Phi(x^-, p^+, \rho, \theta, y) = \sum_m N_m e^{-i(z^- p^+ - x^+ p^+)} e^{im\theta} J_m(\Xi\rho/y) . \quad (5.25)$$

For a given y , there are only two types of CSR: one bosonic and one fermionic corresponding to whether m is integer or half-odd integers respectively.

CHAPTER 6

CONTRACTION OF THE MASSIVE REPRESENTATION IN FIVE DIMENSIONS

In this chapter, we consider the contraction of the five dimensional massive representation based on the paper by Khan and Ramond [22]. The little group is $SO(4)$. Instead of realizing linearly, we choose to construct the generators using three pairs of canonical coordinates and the conjugate momenta, which is a particular non-linear realization of the $SO(4)$ little group. This non-linear realization allows us to use the Majorana infinite component wave theory [23], introduced long ago. It is amusing that the $SO(4)$ symmetry properties of the Majorana model are exactly same as those of the Bohr's non-relativistic hydrogen atom. In the following sections, we show how to construct the non-linear realization, identify the generators consistent with the five dimensional Poincaré generators in Dirac-Bacry-Chang representation, explain the contraction procedure and finally obtain the representation wavefunction. We also compare our result with Wigner's form of the CSR in four dimensions.

6.1 Non-linear Realization and the Majorana Theory

The linear realization of the generators of the $SO(4)$ little group in five dimensions requires four internal canonical coordinates and their conjugate momenta which satisfy the commutation relations,

$$[\eta^i, \pi^j] = i\delta^{ij}, \quad i, j = 1, 2, 3. \quad \text{and} \quad [\eta^4, \pi^4] = i. \quad (6.1)$$

To realize non-linearly, we express the fourth coordinate and its conjugate momentum in terms of the remaining variables as

$$\eta^4 = -\frac{1}{M}(\eta^i \pi^i - i) \quad \text{and} \quad \pi^4 = -\frac{1}{M}\left(\pi^i \pi^i - \frac{\mu}{\eta}\right), \quad (6.2)$$

where M is the mass, μ is an internal parameter with the dimension of mass and $\eta = \sqrt{\eta^i \eta^i}$. Using these, the generator S^{4i} can be expressed as

$$\begin{aligned} S^{4i} &= \eta^4 \pi^i - \eta^i \pi^4, \\ &= \frac{1}{M} \left(-\eta^j \pi^j + \eta^i \pi^i + i\pi^i - \frac{\mu \eta^i}{\eta} \right), \\ &= \frac{1}{M} \left(\pi^j S^{ij} - \frac{\mu \eta^i}{\eta} - i\pi^i \right). \end{aligned} \quad (6.3)$$

Using $S^{ij} = \epsilon^{ijk} S^k$, we find,

$$MS^{4i} = \epsilon^{ijk} \pi^j S^k - \frac{\mu \eta^i}{\eta} - i\pi^i, \quad (6.4)$$

which is exactly the same form as the Laplace-Runge-Lenz (LRL) vector and it satisfies the following commutation relations,

$$[MS^{4i}, MS^{4j}] = iM^2 S^{ij}, \quad (6.5)$$

$$[S^{ij}, MS^{4k}] = i(\delta^{ik} MS^{4j} - \delta^{jk} MS^{4i}). \quad (6.6)$$

This allows us to relate the mass M with the Bohr Hamiltonian,

$$M = \sqrt{-2\mu H}, \quad (6.7)$$

where H , the Bohr Hamiltonian, is given by

$$H = \frac{\pi^i \pi^i}{2\mu} - \frac{1}{\eta}. \quad (6.8)$$

So the parameter μ is the mass of the electron with the electric charge set to one.

On the other hand, the generators of the massive little group in five dimensions satisfy the commutation relations,

$$[S^{ij}, T^k] = i(\delta^{ik} T^j - \delta^{jk} T^i), \quad (6.9)$$

$$[T^i, T^j] = iM^2 S^{ij}. \quad (6.10)$$

The commutators in Eqs.(6.5-6.6,6.9-6.10) suggest to identify the light-cone vector T^i with the LRL vector MS^{4i} . Therefore we can use the mathematics of the hydrogen atom to find the spectrum of the light-cone little group. In this nonlinear realization, the light-cone Hamiltonian and the boost generators become

$$P^- = \frac{p^i p^i + M^2}{2p^+} = \frac{p^i p^i - 2\mu H}{2p^+}, \quad (6.11)$$

$$M^{-i} = x^- p^i - \frac{1}{2} \{x^i, P^-\} + \frac{1}{p^+} (T^i - p^j S^{ij}). \quad (6.12)$$

Since $SO(4) \sim SO_+(3) \times SO_-(3)$, the commuting generators of $SO_\pm(3)$ group can be identified with the combinations

$$\frac{1}{2} \left(S^k \pm \frac{T^k}{\sqrt{-2\mu H}} \right).$$

Since

$$S^i T^i = T^i S^i = 0,$$

the two Casimir eigenvalues of the $SO(4)$ are same which is given by

$$C_2 = \frac{1}{4} \left(S^k S^k - \frac{T^k T^k}{2\mu H} \right) = j(j+1) , \quad (6.13)$$

where j is the eigenvalue of the $SO_{\pm}(3)$ angular momentum algebra. Therefore the eigenvalue of the Bohr Hamiltonian is

$$H = - \frac{\mu/2}{(2j+1)^2} ,$$

and so the mass M can be expressed in terms of j ,

$$M_j^2 = \frac{\mu^2}{4C_2 + 1} = \frac{\mu^2}{(2j+1)^2} . \quad (6.14)$$

So at each mass level, the states assemble in a representation of $SO(4)$ and generate the spectrum of the infinite component Majorana theory.

We now apply contraction to this model. Since $SO(4) \supset SO(3) \supset SO(2)$, we need to contract at least twice to obtain the $E(2)$ representation of the little group in four dimensions. There are two ways to proceed:

1. We contract, staying in five dimensions first, and then use KK reduction and contract to four dimensions, i.e.,

$$SO(4) \xrightarrow[\text{contraction}]{1\text{st}} E(3) \xrightarrow[\text{reduction}]{\text{KK}} SO(3) \xrightarrow[\text{contraction}]{2\text{nd}} E(2) .$$

2. We apply the KK reduction first and then contract, i.e.,

$$SO(4) \xrightarrow[\text{reduction}]{\text{KK}} SO(3) \xrightarrow[\text{contraction}]{\text{Double}} E(2) .$$

In the following two sections, we find the conditions to obtain the CSR in four dimensions for these two cases.

6.1.1 $SO(4) \rightarrow E(3) \rightarrow SO(3) \rightarrow E(2)$

The states are eigenstates of the diagonal subgroup which is the angular momentum, and given by the sum of $SO_+(3)$ and $SO_-(3)$ groups of $SO(4)$,

$$S^i S^i = s(s+1) , \quad (6.15)$$

where $s = 0, 1, 2, \dots, 2j$. Therefore each mass level is $(2s+1)$ -fold degenerate. The contraction scenario is very similar to the IW contraction process $SO(3) \rightarrow E(2)$ where m is kept fixed while j goes to infinity. In this case, the contraction process for $SO(4) \rightarrow E(3)$ is to take j , the quantum number for $SO(4)$, to infinity while the quantum number s for $SO(3)$ remains finite. Since M_j and

j are inversely related, the $j \rightarrow \infty$ limit leads to zero mass and commuting light-cone vectors,

$$M_j = \frac{\mu}{2j+1} \rightarrow 0, \quad \text{and so} \quad [T^i, T^k] \rightarrow 0, \quad (6.16)$$

and thus the generator S^{ij} and T^i form the $E(3)$ algebra. Note that this contraction process do not change the space-time dimensions. This is the CSR in five dimensions.

To find the Casimir of the $E(3)$ group, we multiply the $SO(4)$ Casimir by a contraction parameter ϵ ,

$$(\epsilon S^i)(\epsilon S^i) + \frac{\epsilon^2}{M_j^2} T^i T^i = 4 \epsilon^2 j(j+1). \quad (6.17)$$

As $\epsilon \rightarrow 0$ and $j \rightarrow \infty$, such that ϵj remains fixed, we get

$$T^i T^i = \mu^2. \quad (6.18)$$

That is, the length of the translation vector is fixed, and this is a Casimir of the $E(3)$ group. The other Casimir is given by the zero form,

$$W = \epsilon_{ijk} T^i S^{jk} = T^4 S^i = 0. \quad (6.19)$$

Comparing to Eq.(2.10) in Sec.2.2.2, we readily see that these are the Poincaré Casimirs and the remaining Casimir is of course the mass squared which is zero. These Casimirs uniquely determine the representation in five dimensions.

The above analysis also indicates that we could have chosen μ or $1/j$ as the contraction parameter, because both of these lead the mass operator to zero. However, if $\epsilon = \mu$, then following Eq.(6.18), we find null light-cone vector which yields the regular massless representation, not the CSR. The other choice $1/j$ does not change the above result at all. Therefore, $1/j$ is actually the contraction parameter.

After the first contraction, the light-cone hamiltonian reduces to $P^- = \frac{p^i p^i}{2p^+}$, and the boost generators do not change at all. The KK reduction changes these two generators to

$$P^- = \frac{p^a p^a + M_n}{2p^+}, \quad (6.20)$$

$$M^{-a} = x^- p^a - \frac{1}{2} \{x^a, P^-\} + \frac{1}{p^+} (\hat{T}^a - p^b S^{ab}), \quad (6.21)$$

where $a, b = 1, 2$, and $M_n = \frac{n}{R}$. The new light-cone vector is now

$$\hat{T}^a = T^a + \frac{n}{R} S^{3a}. \quad (6.22)$$

The generators $\hat{T}^a/M_n$ and S^{12} form the massive $SO(3)$ little group. Let $\bar{j}$ be the eigenvalue of the KK reduced $SO(3)$ algebra. It is now quite clear by now that to obtain a CSR in four dimension, we must take the limits, $\bar{j} \rightarrow \infty$ and $R \rightarrow \infty$ such that the ratio $\frac{\bar{j}}{R}$ remains finite and nonzero,

$$\hat{T}^a \hat{T}^a = \frac{n^2 \bar{j}^2}{R^2} \equiv \bar{\Xi}^2 = \text{finite} . \quad (6.23)$$

This is the Casimir of the little group in four dimensions which together with the zero mass-squared uniquely label the representation. We find the representation wavefunction in Sec.6.2.

6.1.2 $SO(4) \rightarrow SO(3) \rightarrow E(2)$

This sequence is a two-step process to obtain the light-cone little group $E(2)$ in four dimensions from the $SO(4)$ little group in five dimensions: first we compactify along the third space (orbital) direction by making it periodic. The light-cone Hamiltonian becomes

$$P^- = \frac{1}{2p^+} (p^a p^a + M_{jn}^2) ,$$

for $a = 1, 2$, with

$$M_{jn}^2 = \frac{\mu^2}{(2j+1)^2} + \frac{n^2}{R^2} , \quad (6.24)$$

where we used $p^3 = n/R$ (momentum along the third direction). For finite j and R , there are no massless states, and the massive little group is $SO(3)$ in four dimensions, generated at each mass level by

$$S^3 \quad \text{and} \quad \frac{1}{M_{jn}} (T^a + \frac{n}{R} S^{3a}), \quad a = 1, 2.$$

deduced by looking at M^{-a} . The spectrum is made up of massive particles with spin j' , which depends on j and n through the quadratic Casimir

$$(S^3)^2 + \frac{1}{M_{jn}^2} \left(T^a + \frac{n}{R} S^{3a} \right) \left(T^a + \frac{n}{R} S^{3a} \right) = j'(j' + 1) . \quad (6.25)$$

Eq.(6.24) clearly shows that a necessary condition for the mass operator to vanish is to take the double singular limit where both j and R tend to infinity. But this is not enough because the following condition

$$\hat{T}^1 \hat{T}^1 + \hat{T}^2 \hat{T}^2 \equiv \bar{\Xi}'^2 < \infty ,$$

must satisfy to yield the CSR. This requires, in addition to the double limit, the ratio $\frac{j'}{R}$ to remain finite and nonzero. To see it explicitly, divide Eq.(6.25) by R^2 and rearranging,

$$\hat{T}^a \hat{T}^a = \left(\frac{\mu^2}{(2j+1)^2/R^2} + n^2 \right) \left(\frac{j'^2}{R^2} - \frac{m^2}{R^2} \right) ,$$

where we used $S^3 = m$. The double limit certainly does not give any nonzero value, but the following triple limit

$$j, j', R \rightarrow \infty, \quad \text{such that} \quad \frac{j}{R}, \frac{j'}{R} \quad \text{remain finite (non-zero)},$$

does give a finite value. Therefore, in this triple limit, we find,

$$(\hat{T}^a)^2 = \left(\frac{\mu^2}{(2j/R)^2} + n^2 \right) \left(\frac{j'}{R} \right)^2 \equiv \Xi'^2, \quad (6.26)$$

which is the Casimir in four dimensions. Together with the mass squared (which is zero) it uniquely labels the representation. This is what we expected, because the generator S^{12} was not affected by any of the reduction and/or contraction processes. The contraction only affected the quadratic Casimir eigenvalue which sets the range of quadratic Casimir of its maximal subgroup. The above analysis suggest how to identify the contraction parameter and the required singular limits to obtain any CSR: look at the mass operator and find what limits provide zero mass and if these limits keep the light-cone vector non-zero and finite. Unlike the previous section, the contraction parameter has a physical meaning which is the inverse KK radius.

6.2 Representation of the Wavefunction

Since we have the form of the T^i 's in terms of the internal variables, we can explicitly find the representation of the wave function for the light-cone vectors $\hat{T}^a$. First note that in $\hat{T}^a$ the contraction parameter j is not explicitly present. Therefore if we apply the contraction it will only include the $R \rightarrow \infty$ limit, not the $j \rightarrow \infty$. As a result we will not obtain commuting vectors which is required to obtain the CSR. Since the contraction parameter j appears through the Hamiltonian when it acts on the wavefunction, we rewrite T^i in terms of the Hamiltonian as in the following,

$$\begin{aligned} T^i &= \epsilon_{ijk} \pi^j L^k - \frac{\mu \eta^i}{\eta} - i \pi^i, \\ &= \eta^i \left(2\mu H + \frac{\mu}{\eta} \right) - (\eta \cdot \pi - i) \pi^i, \end{aligned}$$

where the Hamiltonian is given in Eq.(6.8). So the light-cone vector, $\hat{T}^a$, can be expressed as

$$\hat{T}^a = \left(2\mu \eta^a H + \frac{\mu \eta^a}{\eta} - \frac{n}{R} \eta^a \pi^3 - \mu [\eta^a, H] \right) + (y + 2i - \eta \cdot \pi) \pi^a, \quad (6.27)$$

where $y = n\eta^3/R$. The first term is dependent on the contraction parameters whereas the second term is independent and also both terms are separately hermitian. Since $H \sim \mathcal{O}(1/j^2)$ and $1/\eta \sim \mathcal{O}(1/R)$, the first term vanishes as $j, R \rightarrow \infty$. It should be noted that if we evaluate the commutator in the first term in Eq.(6.27) and then consider the contraction, the resulting light-cone vector becomes non-hermitian and the whole analysis becomes physically meaningless. It is not well

understood why this happens, but to maintain the hermiticity, we have to consider the contraction without computing the commutator. Using

$$\eta \cdot \pi = \eta^a \pi^a - iy \frac{\partial}{\partial y} ,$$

and dropping the first term, the light-cone vector becomes

$$\hat{T}^a = \left(y + 2i + iy \frac{\partial}{\partial y} - \eta^b \pi^b \right) \pi^a . \quad (6.28)$$

It can be easily checked that the above form satisfies the $E(2)$ algebra,

$$[\hat{T}^a, \hat{T}^b] = 0 \quad \text{and} \quad [S^{ab}, \hat{T}^c] = i \left(\delta^{ac} \hat{T}^b - \delta^{bc} \hat{T}^a \right) ,$$

which is required to obtain any CSR in four dimensions. We now find the representation of the wavefunction for this $E(2)$ algebra.

The square of the light-cone vector is,

$$(\hat{T}^a)^2 = \left[(\eta^b \pi^b)^2 - i\eta^b \pi^b \left(-2iy + 5 + 2y \frac{d}{dy} \right) - \left(y^2 \frac{d^2}{dy^2} - (2iy^2 - 6y) \frac{d}{dy} - (y^2 + 6iy - 6) \right) \right] (\pi^2) ,$$

On a wavefunction of the form,

$$\Phi(\eta^i) = e^{iy} \phi(\eta^a, y) ,$$

this gives the following form of the differential equation,

$$\left[(\eta^b \pi^b)^2 - i\eta^b \pi^b \left(5 + 2y \frac{d}{dy} \right) - \left(y^2 \frac{d^2}{dy^2} + 6y \frac{d}{dy} + 6 \right) \right] (\pi^2) = \Xi^2 , \quad (6.29)$$

where $\Xi = \tilde{\Xi}$ or Ξ' corresponding to Cases 1 and 2 respectively. To solve the above equation, let

$$(\pi^a)^2 = \Pi^2(y) \quad (6.30)$$

when it acts on $\phi(\eta^a, y)$, that is the representation of the η^a dependent part of the wave function is the Bessel function, namely,

$$\phi(\eta^a, y) = J_m(\Pi(y)\rho) e^{im\theta} F(y) ,$$

where $F(y)$ is a real function. In polar coordinates ($\eta^1 = \rho \cos \theta$ and $\eta^2 = \rho \sin \theta$), we use,

$$-i\eta^a \pi^a = -\rho \frac{d}{d\rho} \quad \text{and} \quad (\eta^a \pi^a)^2 = -\left(\rho^2 \frac{d^2}{d\rho^2} + \rho \frac{d}{d\rho} \right) ,$$

and rewrite Eq.(6.29) as

$$\left[\rho^2 \frac{d^2}{d\rho^2} + \left(6 + 2y \frac{d}{dy} \right) \rho \frac{d}{d\rho} + \left(y^2 \frac{d^2}{dy^2} + 6y \frac{d}{dy} + 6 \right) \right] (\Pi^2 J_m F) = -\Xi^2 J_m F . \quad (6.31)$$

Making use of the following identities,

$$\frac{d}{d\rho} J_m(\Pi(y)\rho) = \frac{\Pi}{2} (J_{m-1} - J_{m+1}) , \quad (6.32)$$

$$\frac{d}{dy} J_m(\Pi(y)\rho) = \frac{\rho}{2} \frac{d\Pi}{dy} (J_{m-1} - J_{m+1}) , \quad (6.33)$$

and after straightforward algebra, the left-hand side of Eq.(6.31) can be expressed as a linear combination of Bessel function of different orders,

$$\begin{aligned} & \left(\frac{\rho^2 \Pi^4 F}{4} + \frac{\rho^2 y \Pi^3 \Pi' F}{2} + \frac{\rho^2 y^2 \Pi^2 \Pi'^2 F}{4} \right) (J_{m-2} + J_{m+2}) \\ & + \left(3\rho \Pi^3 F + 6\rho y \Pi^2 \Pi' F + \rho y \Pi^3 F' + 2\rho y^2 \Pi \Pi'' F \right. \\ & \quad \left. + \rho y^2 \Pi^2 \Pi' F' + \frac{\rho y^2 \Pi^2 \Pi'' F}{2} \right) (J_{m-1} - J_{m+1}) \\ & + \left(-\frac{\rho^2 \Pi^4 F}{2} - \rho^2 y \Pi^3 \Pi' F + 2y^2 \Pi'^2 F + 2y^2 \Pi \Pi'' F + 4y^2 \Pi \Pi' F' \right. \\ & \quad \left. + y^2 \Pi^2 F'' - \frac{\rho^2 y^2 \Pi^2 \Pi'^2 F}{2} + 12y \Pi \Pi' F + 6y \Pi^2 F' + 6\Pi^2 F \right) J_m , \end{aligned}$$

where the 'prime' denotes derivative with respect to the argument which are suppressed for simplicity. By matching the coefficients of Bessel functions of different order with the right-hand side of Eq.(6.31), we get three constraints. Equating the coefficient of $J_{m\pm 2}$ to zero gives the first constraint,

$$\frac{\rho^2 \Pi^2}{4} \left(\Pi^2 + 2y \Pi \frac{d\Pi}{dy} + y^2 \left(\frac{d\Pi}{dy} \right)^2 \right) F(y) = 0 ,$$

which is satisfied if

$$\Pi^2 + 2y \Pi \frac{d\Pi}{dy} + y^2 \left(\frac{d\Pi}{dy} \right)^2 = 0 . \quad (6.34)$$

The solution to this equation is given by

$$\Pi(y) = \frac{\Pi_0}{y} , \quad (6.35)$$

where Π_0 is a constant to be determined by the boundary condition. The constraint obtained by equating the coefficients of $J_{m\pm 1}$ provides no new result. Finally the remaining constraint is from the coefficients of J_m which gives a differential equation for $F(y)$,

$$\frac{d^2 F}{dy^2} + \frac{2}{y} \frac{dF}{dy} + \frac{\Xi^2}{\Pi_0^2} F = 0 , \quad (6.36)$$

where we used $\Pi(y) = \Pi_0/y$ and its derivatives. The solution is

$$F(y) = \frac{1}{y} \left(A \sin \left(\frac{\Xi y}{\Pi_0} \right) + B \cos \left(\frac{\Xi y}{\Pi_0} \right) \right),$$

where A and B are constants. Since $F(y)$ is regular at $y = 0$, we set $B = 0$. Therefore the wave function can be written as

$$\Phi(\rho, \theta, y) = \sum_m N_m \left(\frac{e^{iy}}{y} \sin \left(\frac{\Xi y}{\Pi_0} \right) \right) e^{im\theta} J_m \left(\frac{\Pi_0 \rho}{y} \right), \quad (6.37)$$

where N_m is the overall normalization constant.

In their paper [9], IW found the following wavefunction for the $E(2)$ algebra,

$$\Phi_{IW}(\rho, \theta) = \sum_m A_m e^{im\theta} J_m(\Xi \rho), \quad (6.38)$$

where Ξ^2 , the square of the Pauli-Lubanski vector, is the second Casimir of the Poincaré group, and labels the CSR in four-dimensions. The Euclidean vector is linear in π^a and as a result its length appeared as a scale factor in the Bessel function. In four dimensions there are only two type of CSRs, fermionic and bosonic types corresponding to half-odd and integer values of m . The amplitude is also constant.

Although similar in form, there are differences between Eq.(6.37) and the IW form in Eq.(6.38). In our case, the internal momentum (Eq.(6.35)) and amplitude are not constant, but functions of y which is the ratio of the internal and external coordinates. It is due to the fact that the Euclidean light-cone vector $\hat{T}^a$ is not linear in π^a (in addition to the y dependence). Moreover, unlike IW case, the length of the light-cone vector does not appear as a scale factor in the Bessel function, even though it is the CSR. To find how our CSR is related to IW's CSR, we must find a relation between Π and Ξ . Let us assume $\Pi = \Xi = \Pi_0/y$. The only solution we get, following Eq.(6.34), is $\Xi = 0$ which corresponds to the regular massless representation, not the CSR. Therefore $\Pi \neq \Xi$ is the only possibility and these are quite new. On the other hand, we may assume that, instead of Π , $\Pi_0 = \Xi$. Substituting this into Eq.(6.37) and setting $y = 1$, the solution becomes exactly the same as that of Wigner's CSR apart from the overall phase factor which has no physical effect (the overall constant factor can be absorbed into the normalization factor). For any other values of y , we have a different kind of CSR because of the non-equality between the scale factor and length of the Euclidean vector. There is no physical reason for Π_0 and Ξ to be equal and $y = 1$, but this is the only condition to obtain IW's result.

Finally, the raising and lowering operators are defined as

$$\hat{T}^{\pm} = \hat{T}^1 \pm i\hat{T}^2 .$$

In polar coordinates, these become

$$\hat{T}^+ = e^{i\theta} \left(2 + y \frac{d}{dy} + \rho \frac{d}{d\rho} \right) \left(\frac{d}{d\rho} + \frac{i}{\rho} \frac{d}{d\theta} \right) , \quad (6.39)$$

$$\hat{T}^- = e^{-i\theta} \left(2 + y \frac{d}{dy} + \rho \frac{d}{d\rho} \right) \left(\frac{d}{d\rho} - \frac{i}{\rho} \frac{d}{d\theta} \right) , \quad (6.40)$$

which acts on $\phi(\rho, \theta, y)$. It is quite obvious that the states $\hat{T}^+|>$ and $\hat{T}^-|>$ have helicities $(m+1)$ and $(m-1)$ respectively. The remaining generator T^3/R becomes

$$\frac{T^3}{R} = (\pi^a)^2 ,$$

under contraction. The complete Poincaré wavefunction is

$$\Phi(x^+, x^a; \rho, \theta, y) = \sum_m N_m \frac{e^{iy}}{y} \sin\left(\frac{\Xi y}{\Pi_0}\right) e^{-i(x^- p^+ - x^a p^a)} e^{im\theta} J_m\left(\frac{\Pi_0 \rho}{y}\right) .$$

We do not understand what is the physical interpretation of the parameter y . But being a ratio of the internal and external coordinates, definitely it relates the internal structure to the external. This may help to understand better the spectrum of the higher dimensional theory.

CHAPTER 7 CONCLUSION

In this dissertation, We have studied two things: first, how the CSRs and the states are characterized in any dimensions with and without supersymmetry and, second, how these representations can be obtained from larger groups by group contraction.

We have shown that in any dimensions, the CSR is characterized by the length of the light-cone vector and the Dynkin label of the short little group which leaves the light-cone vector invariant. We have explicitly shown that, unlike in four dimensions, there are infinitely many CSRs of both bosonic and fermionic types in five dimensions. The states require one more label Ω_{d-3} , the solid angle in d dimensions, which specifies the orientation of the light-cone vector. In the supersymmetric case, we have considered the light-cone vector to be non-nilpotent and nilpotent. In the case of non-nilpotent light-cone vector, it commutes with supercharge. The supermultiplet is like ordinary (zero length light-cone vector) supermultiplet, except that each bosonic and fermionic partner is actually the CSR. In five dimensions, the supermultiplet is like $N = 2$ CSR supermultiplet in four dimensions. The spinorial generators do not have to change their forms. When the light-cone vector is nilpotent, however, we obtain a finite dimensional representation, because the higher order of T^i except T^2 is zero. To make it work we have to introduce more Grassmann parameters, and hence there is no $N = 1$ supersymmetry if T^i is nilpotent. To get both T^i and T^2 nonzero, we need two sets of Grassmann parameters. Introducing more Grassmann parameters will give rise to higher N supersymmetry. The nilpotent construction remedied the infinite heat capacity problem of Wigner, but at the expense of introducing the central charge even though the representation is massless. We have shown this in four and ten dimensions. The nilpotent constructions do not generalize to odd dimensions.

To obtain CSRs from higher dimensions, we need group contraction. First we have shown that the IW group contraction is equivalent to imposing the periodic boundary condition and taking the singular limits. The different between our and Wigner's formalism is in the interpretation of the contraction parameter. In our case, it is the inverse KK radius. We have applied this technique to both five dimensional regular massless and massive representations.

In the regular massless case, we have found that the double limit, $j \rightarrow \infty$ and $R \rightarrow \infty$, yields the CSR if the ratio $\frac{j}{R}$ is nonzero finite and for non-zero mode. For zero mode, the representation is always regular massless in four dimensions. The representation wavefunction is the Bessel function, and the scale factor now includes a parameter y which is a ratio of the internal and external coordinates. This bears the signature of the higher dimension.

In the massive case, we use the infinite component Majorana theory as our model which can be thought of the non-linear realization of the homogeneous $SO(4)$ little group in internal space. We have found that a simple double limit, like in the regular massless case, does not yield any CSR even though the mass term vanishes. We need a triple limit to obtain any CSR in four dimensions. The representation wave function is the Bessel function, but the scale factor does not include the length of the light-cone vector. In fact, the amplitude and scale factor are implicit functions of the parameter y . This is the key difference with Wigner's solution. At $y = 1$ and if $\Pi_0 = \Xi$, our solution and that of Wigner become identical, even though there is no physical reason for these conditions. For any other value of y , the solutions are quite new.

Our analysis of the contraction of the five dimensional algebra suggests how to identify the contraction parameter correctly which will lead to the Euclidean algebra, essential for the existence of the CSR. We need to look at the mass term first and find the parameters which will make the mass term zero, and then look at the quadratic Casimir of the non-contracted group and find if this parameter leads to non-zero length; if not it is not the CSR. This procedure obviously generalizes to higher dimensions.

APPENDIX A CSR AND THE CONFORMAL GROUP

The conformal group is one of the most familiar group which contain the Poincaré group by embedding. In d -dimension, the $SO(d, 2)$ Conformal group include the Dilatation and Conformal translation generators, in addition to the Poincaré generators. It has been known [7, 8] for long time that only the regular massless representation of the Poincaré group has unique Conformal extension. But there is no elegant proof for that. In this appendix, we proof this in light-cone language.

We express the generators of the Conformal algebra using the same variables as for the Poincaré algebra. We add to the Poincaré generators, the dilatation generator

$$D = \frac{1}{2} (x \cdot p + p \cdot x) , \quad (\text{A.1})$$

for the scale transformation and the special conformal transformations generated by

$$K^\mu = 2x_\nu M^{\mu\nu} + x^2 p^\mu = 2x^\mu D - x^2 p^\mu + 2x^\nu S^{\mu\nu} . \quad (\text{A.2})$$

where $\mu = 0, 1, \dots, (d-1)$, and we used,

$$M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu + S^{\mu\nu} . \quad (\text{A.3})$$

In addition to the commutation relations of the Poincaré algebra, these two generators satisfy the following commutation relations,

$$[M^{\mu\nu} , D] = 0 , \quad [D , p^\mu] = i p^\mu , \quad [D , K^\mu] = -i K^\mu ,$$

$$[M^{\mu\nu} , K^\alpha] = i (\eta^{\mu\alpha} K^\nu - \eta^{\nu\alpha} K^\mu) , \quad [p^\mu , K^\nu] = -2i (\eta^{\mu\nu} D - M^{\mu\nu}) .$$

To obtain the light-cone forms of the Dilatation and the Conformal translation generators, we first express the generators in light-cone coordinates and apply boost along the $(d-1)$ -direction,

$$D = \frac{1}{2} (-\{x^-, p^+\} - \{x^+, p^-\} + \{x^i, p^i\}) , \quad (\text{A.4})$$

$$K^+ = -2x^+ M^{+-} + x^2 p^+ + 2x^i M^{+i} , \quad (\text{A.5})$$

$$K^- = 2x^- M^{+-} + 2x^i M^{-i} + x^2 p^- , \quad (\text{A.6})$$

$$K^i = 2x^- M^{+i} + 2x^+ M^{-i} + 2x^j M^{ij} + x^2 p^i ,$$

where $i, j = 1, \dots, (d-2)$. Using the light-cone forms of the Poincaré generators from Eqs.(2.5,2.7) and setting $x^+ = 0$, K^- and K^i can be written as

$$\begin{aligned} K^- &= 2x^- (-x^- p^+ + x^i p^i) - x^i x^i p^- + \frac{2}{p^+} (x^i T^i - x^i p^j S^{ij}) , \\ K^i &= 2x^i (-x^- p^+ + x^i p^i) - x^j x^j p^i + 2x^j S^{ij} . \end{aligned}$$

To check if the above forms satisfy the algebra, we calculate the commutators $[P^-, K^i]$ and $[p^i, K^-]$. Substituting the light-cone forms, we calculate

$$[P^-, K^i] = 2i \left(x^- p^i - \frac{1}{2} \{x^i, P^-\} - \frac{p^j}{p^+} S^{ij} \right) , \quad (\text{A.7})$$

$$[K^-, p^i] = 2i \left(x^- p^i - \frac{1}{2} \{x^i, P^-\} + \frac{1}{p^+} (T^i - p^j S^{ij}) \right) . \quad (\text{A.8})$$

On the other hand these two commutators are given by

$$[P^-, K^i] = [K^-, p^i] = 2iM^{-i} , \quad (\text{A.9})$$

where M^{-i} is given in Eq.(2.7). So comparing Eqs.(A.7-A.9), it is clear that we must set $T^i = 0$, to satisfy the Conformal algebra. Therefore, the light-cone forms of the conformal generators are

$$K^+ = -x^i x^i p^+ , \quad (\text{A.10})$$

$$K^i = 2x^i D - x^j x^j p^i + 2x^j S^{ij} , \quad (\text{A.11})$$

$$K^- = 2x^- D - x^i x^i p^- - \frac{2}{p^+} x^i p^j S^{ij} , \quad (\text{A.12})$$

where now the dilatation generator is given by

$$D = \frac{1}{2} (-\{x^-, p^+\} + \{x^i, p^i\}) . \quad (\text{A.13})$$

Since $T^i = 0$, it implies that this is the regular massless representation, not the CSR. Therefore, to obtain the CSR, we must use group contraction which produce the Euclidean group, an essential ingredient for the CSR to exist.

APPENDIX B
ANGULAR MOMENTUM IN CYLINDRICAL COORDINATES WITH
PERIODIC BOUNDARY CONDITION

The cylindrical form of the angular momentum operator is given in Eqs.(4.8-4.10) which acts on the wavefunction of the form as in Eq.(4.6). When these operators act on the wavefunction, we can replace $\frac{\partial}{\partial z}$ by $\frac{in}{z}$. Substituting this, the squares of S^{23} and S^{31} become

$$(S^{23})^2 = -z^2 \left[\sin^2 \theta \frac{\partial^2}{\partial \rho^2} + \frac{\cos^2 \theta}{\rho} \frac{\partial}{\partial \rho} + \frac{2 \sin \theta \cos \theta}{\rho} \frac{\partial^2}{\partial \theta \partial \rho} - \frac{\rho \sin^2 \theta}{z^2} \frac{\partial}{\partial \rho} \right. \\ \left. - \frac{2 \rho \sin^2 \theta}{z} \frac{in}{R} \frac{\partial}{\partial \rho} - \frac{2 \sin \theta \cos \theta}{\rho^2} \frac{\partial}{\partial \theta} + \frac{\cos^2 \theta}{\rho^2} \frac{\partial^2}{\partial \theta^2} - \frac{\sin \theta \cos \theta}{z^2} \frac{\partial}{\partial \theta} \right. \\ \left. - \frac{2 \sin \theta \cos \theta}{z} \frac{in}{R} \frac{\partial}{\partial \theta} - \frac{1}{z} \frac{in}{R} - \frac{\rho^2 \sin^2 \theta}{z^2} \frac{n^2}{R^2} \right], \quad (\text{B.1})$$

$$(S^{31})^2 = -z^2 \left[\cos^2 \theta \frac{\partial^2}{\partial \rho^2} + \frac{\sin^2 \theta}{\rho} \frac{\partial}{\partial \rho} - \frac{2 \sin \theta \cos \theta}{\rho} \frac{\partial^2}{\partial \theta \partial \rho} - \frac{\rho \cos^2 \theta}{z^2} \frac{\partial}{\partial \rho} \right. \\ \left. - \frac{2 \rho \cos^2 \theta}{z} \frac{in}{R} \frac{\partial}{\partial \rho} + \frac{2 \sin \theta \cos \theta}{\rho^2} \frac{\partial}{\partial \theta} + \frac{\sin^2 \theta}{\rho^2} \frac{\partial^2}{\partial \theta^2} + \frac{\sin \theta \cos \theta}{z^2} \frac{\partial}{\partial \theta} \right. \\ \left. + \frac{2 \sin \theta \cos \theta}{z} \frac{in}{R} \frac{\partial}{\partial \theta} - \frac{1}{z} \frac{in}{R} - \frac{\rho^2 \cos^2 \theta}{z^2} \frac{n^2}{R^2} \right], \quad (\text{B.2})$$

and also $(S^{12})^2 = -\frac{\partial^2}{\partial \theta^2}$. Therefore

$$\vec{S}^2 \equiv (S^{ij})^2 = -z^2 \left[\frac{\partial^2}{\partial \rho^2} + \left(\frac{1}{\rho} - \frac{\rho}{z^2} - \frac{2in\rho}{zR} \right) \frac{\partial}{\partial \rho} \right. \quad (\text{B.3})$$

$$\left. + \left(\frac{1}{\rho^2} + \frac{1}{z^2} \right) \frac{\partial^2}{\partial \theta^2} - \frac{2in}{zR} - \frac{\rho^2 n^2}{z^2 R^2} \right]. \quad (\text{B.4})$$

Let s^2 be the eigenvalue of $\vec{S}^2$. The eigenvalue equation now becomes

$$\left[\frac{\partial^2}{\partial \rho^2} + \left(\frac{1}{\rho} - \frac{\rho}{z^2} - \frac{2in\rho}{zR} \right) \frac{\partial}{\partial \rho} + \left(\frac{1}{\rho^2} + \frac{1}{z^2} \right) \frac{\partial^2}{\partial \theta^2} \right. \\ \left. + \frac{s^2}{z^2} - \frac{2in}{zR} - \frac{\rho^2 n^2}{z^2 R^2} \right] \Phi_n(\rho, \theta) = 0. \quad (\text{B.5})$$

Using separation of variables, we write

$$\Phi_n(\rho, \theta) = \mathcal{R}_n(\rho) \Theta(\theta), \quad (\text{B.6})$$

and substituting into the eigenvalue equation, the differential equation factors into a radial and an angular part,

$$\left(\frac{1}{\rho^2} + \frac{1}{z^2}\right)^{-1} \left[\frac{1}{\mathcal{R}_n} \frac{\partial^2 \mathcal{R}_n}{\partial \rho^2} + \left(\frac{1}{\rho} - \frac{\rho}{z^2} - \frac{2in\rho}{zR} \right) \frac{1}{\mathcal{R}_n} \frac{\partial \mathcal{R}_n}{\partial \rho} \right. \quad (\text{B.7})$$

$$\left. + \frac{s^2}{z^2} - \frac{2in}{zR} - \frac{\rho^2 n^2}{z^2 R^2} + \frac{1}{\Theta} \frac{\partial^2 \Theta}{\partial \theta^2} \right] = 0. \quad (\text{B.8})$$

Let the solution of the angular part is of the form

$$\Theta(\theta) = e^{im\theta}, \quad (\text{B.9})$$

up to a normalization constant and where m is the azimuthal quantum number. Substituting this, we obtain the differential equation of the radial coordinate only,

$$\begin{aligned} & z^2 \frac{d^2}{d\rho^2} \mathcal{R}(\rho) + \left(\frac{z^2}{\rho} - \rho - \frac{2inz\rho}{R} \right) \frac{d}{d\rho} \mathcal{R}(\rho) \\ & + \left(-m^2 \left(1 + \frac{z^2}{\rho^2} \right) - \frac{\rho^2 n^2}{R^2} + s^2 - \frac{2inz}{R} \right) \mathcal{R}(\rho) = 0. \end{aligned}$$

The most general solution of the above differential equation is

$$\begin{aligned} \mathcal{R}(\rho) = & \rho^{-1} e^{\frac{\rho^2(R+2inz)}{4s^2R}} \left[A M \left(\frac{\sqrt{R}(s^2 - m^2 + 1)}{2\sqrt{4inz + R}}, \frac{m}{2}, \frac{\sqrt{4inz + R}\rho^2}{2z^2\sqrt{R}} \right) \right. \\ & \left. + B W \left(\frac{\sqrt{R}(s^2 - m^2 + 1)}{2\sqrt{4inz + R}}, \frac{m}{2}, \frac{\sqrt{4inz + R}\rho^2}{2z^2\sqrt{R}} \right) \right], \end{aligned}$$

where M and W are Whittaker functions defined in terms of confluent hypergeometric functions as

$$M_{a,b}(x) = e^{-\frac{x}{2}} x^{b+\frac{1}{2}} {}_1F_1(b-a+\frac{1}{2}, 2b+1; x), \quad (\text{B.10})$$

$$W_{a,b}(x) = e^{-\frac{x}{2}} x^{b+\frac{1}{2}} U(b-a+\frac{1}{2}, 2b+1; x), \quad (\text{B.11})$$

where ${}_1F_1$ and U are the confluent Hypergeometric functions of first and second kind respectively.

Their integral representations are given by

$${}_1F_1(a, b; x) = \frac{\Gamma(b)}{\Gamma(b-a)\Gamma(a)} \int_0^1 e^{xt} t^{a-1} (1-t)^{b-a-1} dt, \quad (\text{B.12})$$

$$U(a, b; x) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-xt} t^{a-1} (1+t)^{b-a-1} dt.$$

As x goes to zero, the function U is divergent and ${}_1F_1$ is finite. For physically relevant solution, we need $\mathcal{R}(\rho)$ to be finite and well-defined at the origin and also as $R \rightarrow \infty$. Therefore, we set $B = 0$.

So the solution is given by the Whittaker function M .

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BIOGRAPHICAL SKETCH

I was born in Dhaka, Bangladesh. While in high school, I became more and more interested in science, especially in mathematics and physics. This motivated me to do my undergraduate studies in physics at the University of Dhaka, Bangladesh. In 1997, I went to the University of Cambridge. There I completed the Part III of the Mathematical Tripos in the Department of Applied Mathematics and Theoretical Physics at the University of Cambridge in 1998. At Cambridge, I became more interested in theoretical physics. In 1999, I enrolled in the Ph.D. program of the Department of Physics at the University of Florida, and since 2001 I have studied in the High Energy Theory Group.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.



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